General relativity

An overview

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Table of Contents

1. Principles of General Relativity

Special Relativity (SR) is based on two principles:

1. The ***principle of relativity*** states that the laws of nature should be the same for all observers in inertial frames (defined below).
2. All such observers, upon measuring the speed of light in a vacuum, must find the same result, c = 299,792,458 km/sec.

The first requirement is necessary in order for physics to be coherent. It means that observers in inertial systems use the same equations. Rather than going on incessantly repeating “in an inertial reference frame”, let’s get it done with once and for all by stating:

> SR considers only observers in ***inertial*** reference frames, those which move with constant, unaccelerated velocity relative to one another.

However, this inertial observer can perfectly well observe objects which are accelerating in his frame. We will see what GR has to say about inertial frames at the end of the next section.

The second requirement is a result of rather astounding experimental results. There is no known reason for it, that’s just how it is. Because of the constant speed of light, SR prevents us from considering space and time as being two separate things and explains how they are related and linked into a more global entity, ***spacetime***[1].

General Relativity (GR) builds on top of SR to take into account three additional requirements:

1. The ***principle of covariance*** is an extension of the first requirement of SR, requiring that physical equations be true in all coordinate systems. GR drops the requirement of inertial systems. In order to do this, new mathematical methods are required – tensors.
2. The ***principle of consistency*** states that the equations of GR must give the same result as the classical (Newtonian) equations when applied in the limiting case, e.g., in inertial frames.
3. The ***principle of equivalence*** exists in two or three different versions, but the strongest one is the most interesting, ***Einstein’s equivalence principle***. This states that in a very small frame of reference, no experiment can distinguish between objects in a gravitational field or objects subject to uniform acceleration.

The principle of equivalence leads to new physics and a new understanding of the universe.

1 Spacetime, in one word, sans hyphen.

2. The equivalence principle and curvature

Imagine four cases:

1. An unaccelerated rocket is coasting along in space with its motors off, far from any matter which might give rise to a gravitational field, and with all the portholes covered so that an astronaut can see nothing outside. If the astronaut holds a wrench and opens her hand, the wrench stays next to her hand. In fact, if she drops two wrenches of different weights, they will both undergo the same acceleration, exactly as under the influence of gravity. Both will remain wherever she let go of them.
2. A person in a very high elevator, or an elevator in a very deep mine shaft, also has no view of the exterior. If the elevator cable breaks so that the elevator is falling freely and the person lets go of a wrench, the wrench will remain by his hand.
3. Now consider the elevator as sitting still on the surface of the Earth. If the person in the stationary elevator lets go of the wrench, it is in his interest first to assure that his foot is not just below it, as the wrench will fall to the floor.
4. Meanwhile, back in the rocket, the motors are turned on so as to give an acceleration of 1g to the rocket. If the astronaut releases her wrench, it too will drop to the "floor" of the rocket.

If you don't like the idea of falling elevators (which was Einstein's example), think of the reduced-gravity aircraft (aka the "vomit comet") used for habituating astronauts to near-weightless conditions.

The first two cases illustrate the impossibility of distinguishing between free fall and an absence of a gravitational field; the last two, of distinguishing between a gravitational field from uniform acceleration. GR really starts with the recognition that uniform acceleration is indistinguishable from a gravitational field. Einstein's equivalence principle is a precise statement of this idea.

The above four cases depend on one more very important – indeed, essential – thing. Newton's second law of motion says that the acceleration of a body is proportional to the force on it, the proportionality constant being the object's inertial mass, m_I.

$$F = m_I a$$

Newton's law of gravitation says that the gravitational force on the same object is proportional to the products of the object's gravitation mass, m_G, and the mass, M, of the source of the gravitational field.

$$F = G\, m_G M/r^2.$$

Experiments beginning with Galileo show to a very high degree of accuracy that the acceleration due to gravity is independent of the mass of the object. Equating the force and its effect gives

$$m_I a = G\, m_G M/r^{2.}$$

Galileo's observation means that the m_I and m_G must cancel out and therefor be equal, using the following important fact:

The inertial mass of an object is the same as its gravitational mass: $m_I = m_G$

All of GR depends on this equality. making gravity accelerate all objects the same way, regardless of their mass or composition.[2] But how can gravity know how to pull on such different objects so as to impart the same acceleration to both?[3] It seems that gravity has nothing to do with the object, but is simply a property of space: In that case, what else can it be but curvature?

Consider race cars. At each end of a track, the roadbed is steeply banked to keep the cars from skidding off the road (as are highways, but less so, because motorists are not supposed to drive that fast). The angle and curvature of the track are partly responsible for the cars' remaining on the track. Curving, even at constant linear speed, is a form of acceleration.

2 We can ignore friction, since experiments have shown that feathers and anvils fall at the same rate in a vacuum.

3 I lifted this question from Russell Stannard. *Relativity: A very short introduction*. Oxford: Oxford University Press, 2008. Print.

So curvature – curvature of space – can explain the force of gravity in a far “simpler” way than having gravity do something or other depending on the mass and composition of objects. In that case, it is not even a “force”, simply the curvature of space.[4] As objects not submitted to other forces try to follow a straight-line path in a curved space, they naturally act as if they were accelerated – by gravity! Like the car on the race track.

But what the dickens would a straight line be in a curved space? Well, that is the matter of GR, the theory of gravity. In order to do handle this problem, GR must employ the mathematics of curved spaces called ***Riemann spaces***.

Gravity is present everywhere, maybe not much at great distances from stars and planets, but certainly here on the surface of the earth. So the only way to have an inertial reference frame, relative to which a particle is not accelerating, is to use a frame which is freely falling with the particle, as in our first two cases. And the fact that all particles or objects are accelerated the same way in a gravitational field means that the one freely falling reference frame is good for all objects in that frame, as long as there are no other forces present which distinguish one object from another. The conclusion is:

> An ***inertial reference frame*** is one which objects and observers are falling freely.

Now we must see how to take curvature into account mathematically.

4 Some far-out current hypotheses like string theory suppose more (invisible to us) dimensions to spacetime. The curvature of these spaces could explain the other three forces, weak, strong and electromagnetic.

3. Curvature and the mathematics of GR

GR continues like this:

- Space cannot vary too wildly or we could not calculate in it. So we assume that we can use standard methods of differentiation in a very small area of the space. This assumption leads to the notion of ***manifolds***.
- In order to study curved space, we need to manipulate objects which enable us to write equations that satisfy the requirement that they be valid in all reference frames. Such objects are called ***tensors***.
- The object used to measure the space, the ***metric***, must be modified to become a function of space, not just the constant one of Cartesian coordinates. It must reduce to the Cartesian metric in classical physics.
- In order to have valid differential tensor equations, the classical derivative must be replaced by a tensor called the ***covariant derivative***. In the limit of ordinary physics, this should – and does – reduce to the classical derivative. This allows us to calculate the change in a tensor over an infinitesimal distance of smooth, differentiable curved space.
- The tensor equations of motion must reduce to their classical versions under the conditions of classical physics (the principle of consistency).
- And, of course, the theory must agree with observation – hopefully new ones or ones which did not yet have an adequate explanation. This is indeed the case.

An example of the last requirement: SR says that fast-moving objects have slower clocks relative to those of a stationary observer. GR says that objects in a stronger gravitational field (a more curved space) have clocks which slow down. So objects like GPS satellites, far above the earth, appear to have slower clocks than those on the surface of the earth because they are moving (SR), but faster because they are farther from the source (Earth) of the gravitational field (GR). Both these opposite but unequal effects must be taken into account by GPS software.

3.1. Manifolds

We must be careful now. The curvature of space can change as we move from one point to another. It can change differently around every point and in different directions from the point. In a “flat” Euclidean space we calculate the net effect of two vectors by translating the origin of one to the end of the other. But in a general curved space, there is no direct way to do this, since the vector may change as the coordinates do. Instead, we require that a curved space be such that, for an infinitesimally small distance around a point, the space behaves like an n-dimensional Euclidean space. Then we can translate by a succession of infinitesimal displacements. The totality of all these small regions constitutes a ***manifold***.[5]

A manifold must obey the following conditions:

- In the immediate neighborhood of a point, the space is flat, meaning the first derivative (but not necessarily the second) *of a curve* in the space with respect to displacement in any spatial direction must be zero.
- From one point to another point infinitesimally separated from it, the space is smooth, allowing us to calculate the value of a function from one point to the next. At the next point, although the same two requirements must be satisfied, the shape of the space and, so, the coordinates will have changed.

5 Schutz 2009, 142: “This is the way to think of a manifold: it is a space with coordinates, that locally looks Euclidean but that globally can warp, bend, and do almost anything (as long as it stays continuous.)” On p 144, he adds: “The differentiable manifold itself is ’primitive’: an amorphous collection of points, arranged locally like the points of Euclidean space, but not having any distance relation or shape specified. Giving the metric **g** gives it a specific shape...”

A tensor called the ***metric tensor*** may describe the distance between two neighboring points in a differentiable manifold. If the metric exists and is symmetric, the manifold is a ***Riemannian manifold***. This is the type of manifold used in GR.[6]

In order to pass from a vector or tensor field at one point P to the same field at an infinitesimal distance from P, we must employ a mathematical device called a ***connection***. In GR, this may be of different sorts.[7] We will consider only the covariant derivative.

We can understand why we must use curved spaces by realizing that the surface of the Earth is such a space, in two dimensions. Measurement of the sum of the angles of a triangle on such a surface always gives a result greater than 180°, so the space is non-Euclidean. A saddle-shaped surface will give a sum of the angles of a triangle on the surface less than 180°. Also, only on a flat surface will a rectangular path, defined by four right angles end where it started. And the surface of a sphere is equal to 4π times the square of its radius only in a flat, Euclidean space. So it is possible to measure the geometry of a space.

3.2. Tensors

A ***tensor*** is a geometric object which makes a linear map of other objects, vectors or one-forms or a mixture (product) of the two, into a scalar. A tensor of type $\binom{0}{n}$ makes a linear mapping of n vectors into a scalar. A popular image shows something like a vending machine with n input slots for vectors: Put in the appropriate number of vectors and out pops a scalar!

Similarly, a tensor of type $\binom{m}{0}$ does a linear mapping of m one-forms into a scalar.

We start with single-rank tensors, with a single index (set of components), and define ***vectors***; denoted by one upper index, and ***one-forms*** (or ***dual vectors***), denoted by one lower index.[8] A one-form linearly maps a vector into a scalar and a vector linearly maps a one-form into a scalar (by a scalar product), so each is a tensor. The two are distinguished by the way they transform under a coordinate transformation, as follows in the table.

$\binom{1}{0}$ Vector	$V'^{\alpha} = \dfrac{\partial x'^{\alpha}}{\partial x^{\beta}} V^{\beta} = \Lambda^{\alpha}{}_{\beta} V^{\beta}$ (1)
$\binom{0}{1}$ One-form (or dual vector)	$V'_{\alpha} = \dfrac{\partial x^{\beta}}{\partial x'^{\alpha}} V_{\beta} = \Lambda^{\beta}{}_{\alpha} V_{\beta}$ (2)

Table 1: Vector and one-form transformation properties

We shall soon see that the transformation matrix for one-forms is the inverse of that for vectors.

One of the rules of tensor manipulation stipulates that a given index can exist only once or twice. If it exists twice, there must be one upper and one lower index and they are summed over. This is called the ***Einstein summation convention***.

The set of all possible vectors at a point p in spacetime is called the ***tangent space*** at p and constitutes a ***vector space***. Similarly, the space of all linear maps from the vector space at p to real numbers constitutes the ***dual vector space***, also called the ***cotangent space***.[9] So the cotangent space is the space of all one-forms at a point.[10]

6 I don't know what the justification for this is, but it works.

7 In differential geometry or topology, one speaks of ***affine*** connections, of which a special case is the Levi-Civita connection, which in the case of manifolds reduces to the covariant derivative.

8 In days of yore, when I studied physics, vectors were called contravariant tensors and one-forms, covariant tensors.

9 Carroll (2013), 18.

10 The tangent and cotangent spaces can be the same vector space, the elements of which, vectors in another sense, can be multiplied by scalars and added together to form other members of the same vector space.

3.2.1. Vectors, one-forms (dual vectors) and basis vectors

A vector may be represented as sums of components, coefficients of ***basis vectors*** $\vec{e}_\beta$

$$\vec{A} = A^\alpha \vec{e}_\alpha = A^{\bar{\alpha}} \vec{e}_{\bar{\alpha}}. \tag{3}$$

Then, for the Lorentz transformation Λ,

$$A^{\bar{\alpha}} = \Lambda^{\bar{\alpha}}{}_\beta A^\beta$$

and some creative diddling of summed indexes leads to the transformation rule for the basis vectors[11]:

$$\vec{e}_\alpha = \Lambda^\beta{}_\alpha \vec{e}_\beta \tag{4}$$

So basis vectors transform like one-forms.

A one-form maps a vector into a real number (scalar) and so can be represented as a scalar function of a vector:

$$\tilde{p}(\vec{A}) = \text{real number}$$

A tilde over a letter is used to designate a one-form. The ***components*** of a one-form are the result of mapping of the basis vectors:

$$p_\alpha := \tilde{p}(\vec{e}_\alpha)$$

which leads to

$$\tilde{p}(\vec{A}) = A^\alpha p_\alpha$$

and

$$p_{\bar{\beta}} = \Lambda^\alpha{}_{\bar{\beta}} p_\alpha$$

so components of a one-form transform like basis vectors[12]. This inverse transformation leads to the frame invariance of

$$A^{\bar{\alpha}} p_{\bar{\alpha}} = A^\beta p_\beta.$$

Analogously to $\vec{A} = A^\alpha \vec{e}_\alpha$ for vectors, one-forms are defined in terms of basis one-forms ω by

$$\tilde{p} = p_\alpha \tilde{\omega}^\alpha$$

and these formulae lead to the basis vectors and one-forms satisfying

$$\tilde{\omega}^\alpha(\vec{e}_\beta) = \vec{\delta}^\alpha{}_\beta$$

so the βth component of the αth basis one-form is the Kronecker delta function. This equation may be taken as the definition of the relation between basis vectors and one-forms.

The ***tangent*** to a curve parametrized by λ is a vector

$$t^\beta = \frac{dx^\beta}{d\lambda} = \left(\frac{dx^0}{d\lambda}, \frac{dx^1}{d\lambda}, \frac{dx^2}{d\lambda}, \frac{dx^3}{d\lambda}\right) \tag{5}$$

which undergoes a coordinate transformation as follows

$$t'^\alpha = \frac{dx'^\alpha}{d\lambda} = \frac{dx'^\alpha}{dx^\beta}\frac{dx^\beta}{d\lambda} = \Lambda^\alpha{}_\beta t^\beta$$

like a vector should, hence its being written as t^α or $\vec{t}$.

The ***gradient*** of a scalar field Φ is

11 Examples from Schutz, *op. cit.,* 37.

12 These and following examples from Schutz, *op. cit.*, 37-38.

$$\tilde{d}\phi = \Big(\frac{\partial\phi}{\partial t}, \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\Big) \tag{6}$$

which may be represented more generally as

$$\tilde{d}\phi = \Big(\frac{\partial\phi}{\partial x^0}, \frac{\partial\phi}{\partial x^1}, \frac{\partial\phi}{\partial x^2}, \frac{\partial\phi}{\partial x^3}\Big)$$

and is a one-form. This is seen by using the product (Leibniz) rule, for instance,

$$V_{\bar{\mu}} = \frac{\partial\phi}{\partial x^{\bar{\mu}}} = \frac{\partial x^\nu}{\partial x^{\bar{\mu}}}\frac{\partial\phi}{\partial x^\nu} = \Lambda^\nu{}_{\bar{\mu}} V_\nu$$

like a good one-form.

To note:

- A ***basis vector*** is tangent to a coordinate curve, along which only one coordinate changes.
- A ***basis one-form*** is the gradient of a coordinate surface, where one coordinate is constant.
- Since the bases change with the coordinates (unlike Cartesian coordinates), they are called ***coordinate bases***. They are *not* orthonormal.

Since tensor equations are independent of the coordinates (of the bases), they are true in all coordinate systems if they are true in one. This of course is why they are useful in GR.

3.2.2. Of curves, surfaces and normals

A good example is the change of a scalar field along a path parametrized by a variable λ, the ***directional derivative***:

$$\frac{d\phi}{d\lambda} = \frac{\partial\phi}{\partial x^\mu}\frac{dx^\mu}{d\lambda} = \tilde{d}\phi_\mu t^\mu \tag{7}$$

where $\tilde{d}\phi$ is a one-form, the gradient, mapping a vector, the tangent to the curve, into a scalar which represents the rate of change of ϕ along the direction defined by the vector $\vec{V}$. One can also say that a vector is a tangent to a curve and is the function which returns $d\phi/d\lambda$ when It takes $\tilde{d}\phi$ as an argument.[13]

Some (unfortunately for those with weak eyes) oft-seen notation:

$$\frac{\partial\phi}{\partial x^\mu} = \delta_\mu\phi = \phi_{,\mu} \tag{8}$$

Don't miss that comma in the last subscript.

A one-form is considered ***normal*** to a surface if it maps all vectors tangent to the surface to zero,

$$\tilde{p}(\vec{t}) = 0$$

for all tangent vectors $\vec{t}$.

For a function f(x), assumed "well behaved", the gradient

$$\nabla f(x^\alpha)$$

is the steepest slope at a point. An isosurface

$$f(x^\alpha) = constant$$

defines a set of variables, a level set, of members on the surface. Then the gradient of the function at each point on the surface is normal to the surface. (Think of hills and contour lines.) Also,

$$\nabla f(x^\alpha) \cdot \hat{e}$$

13 Schutz, 122,

is the directional derivative along $\hat{e}$.

3.2.3. Higher-rank tensors

Higher-rank tensors can be formed by ***outer products***

$$X = V \otimes W \tag{9}$$

where $X^{\alpha\beta} = V^{\alpha}W^{\beta}$. In general, a tensor $X = V \otimes W$ transforms like this:

$$T^{\mu_1'..\mu_j'}{}_{\nu_1'...\nu_k'} = \Lambda^{\mu_1'}{}_{\mu_1}...\Lambda^{\mu_j'}{}_{\mu_j}\Lambda^{\nu_1}{}_{\nu_1'}...\Lambda^{\nu_k}{}_{\nu_k'}T^{\mu_1..\mu_j}{}_{\nu_1...\nu_k}.$$

The rank is the total number of one-forms and vectors concerned. As an example, consider two one-forms $\tilde{p}$ and $\tilde{q}$. Then their outer product $\tilde{p} \otimes \tilde{q}$ is a rank two tensor of type $\binom{0}{2}$. Operating on vectors $\vec{A}$ and $\vec{B}$ gives

$$\tilde{p}(\vec{A})\tilde{q}(\vec{B})$$

which we we call $\tilde{f}(\vec{A}, \vec{B})$. The most general tensor can be represented as a sum of outer products. For our rank-2 tensor with components

$$f_{\alpha\beta} := \tilde{f}(\vec{e}_{\alpha}, \vec{e}_{\beta}),$$

we can define basis $\binom{0}{2}$ tensors

$$\tilde{\omega}^{\alpha\beta} = \tilde{\omega}^{\alpha} \otimes \tilde{\omega}^{\beta}$$

and show that

$$\tilde{f} = f_{\alpha\beta}\tilde{\omega}^{\alpha} \otimes \tilde{\omega}^{\beta}.$$

3.3. The metric

In SR, the square of the invariant infinitesimal distance is given by

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \tag{10}$$

which could be written as

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \tag{11}$$

where the metric $\eta_{\mu\nu}$ is a diagonal matrix with eigenvalues (-1,1,1,1)[14] in Minkowski space. SR also defines the square of the ***proper time*** as

$$d\tau^2 := -ds^2 \tag{12}$$

In GR, this is generalized to a ***metric*** $g_{\mu\nu}$, which generally is not diagonal and has non-constant components. It may be written as a matrix or a rank 2 one-form. It has an inverse and may be used to raise or lower indices, as shown now and in Table (2).

In terms of tensors, the general ***metric tensor*** g is defined by the dot product of two vectors as

$$\mathbf{g}(\vec{A}, \vec{B}) := \vec{A} \cdot \vec{B} \tag{13}$$

which is symmetric by its definition. The metric is a tensor of type $\binom{0}{2}$ and converts two vectors into a scalar. Also,

$$\mathbf{g}(\vec{A},) := \tilde{V}() \tag{14}$$

converts a single vector into a scalar and is therefore a one-form. So we can write

$$V_\alpha := \tilde{V}(\vec{e}_\alpha) = \vec{V} \cdot \vec{e}_\alpha = \vec{e}_\alpha \cdot (V^\beta \vec{e}_\beta) = (\vec{e}_\alpha \cdot \vec{e}_\beta) V^\beta = \eta_{\alpha\beta} V^\beta$$

illustrating how the metric can be used to lower indices in a tensor equation. The opposite is also true. Notice that in SR, this amounts simply to changing the sign of the zero-th element.

Note that equation (13) defines the inner product of two vectors. In its simplest form in Cartesian coordinates in Euclidean space,

$$\vec{A} \cdot \vec{B} = g_{\mu\nu} A^\mu B^\nu \tag{15}$$

it is clear how g inverts the sign of A^0 on lowering its indices, so that the signature is what we expect, namely, -1,1,1,1.

In GR, the use of a metric is possible because we restrict ourselves to Riemannian manifolds. In this case, the metric g exists and can also be used for raising and lowering of indices.

$$V_\alpha = g_{\alpha\beta} V^\beta.$$

Metric	$\eta_{\mu\nu}$
Inverse metric	$\eta^{\mu\nu}$
Incremental distance and proper time	$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -d\tau^2$
Lowering of vector index (map vector into one-form)	$V_\mu = \eta_{\mu\nu} T^\nu$
Raising of one-form index (map one-form into vector)	$V^\mu = \eta^{\mu\nu} T_\nu$

Table 2: Uses of the metric with tensors

14 Also known as the signature of the metric.

3.4. The covariant derivative

In a curved space, any displacement takes us from one tangent space to another with different basis vectors. Changes in a vector's components therefore have two sources: the vector field itself and changes in the basis vectors, due to the curvature of space. We need to take the variation due to curvature into account in any differentiation. Using differentiation by parts on

$$V = V^\alpha e_\alpha,$$

we get for the β-th component

$$\frac{\partial}{\partial x^\beta} V^\alpha e_\alpha = \frac{\partial V^\alpha}{\partial x^\beta} e_\alpha + V^\alpha \Big(\frac{\partial e_\alpha}{\partial x^\beta} \Big)$$

Defining

$$\frac{\partial e_\alpha}{\partial x^\beta} = \Gamma^\mu_{\alpha\beta} e_\mu \tag{16}$$

meaning that $\Gamma^\mu_{\alpha\beta}$ is the μth component of $\frac{\partial e_\alpha}{\partial x^\beta}$ (its component along the e_μ direction), we can diddle indexes a little to get the ***covariant derivative***

$$\nabla_\beta V^\alpha = \partial_\beta V^\alpha + V^\mu \Gamma^\alpha_{\mu\beta} \tag{17}$$

or, in yet another notation,

$$\frac{DV^\alpha}{Dx^\beta} = \frac{\partial V^\alpha}{\partial x^\beta} + \Gamma^\alpha_{\mu\beta} V^\mu$$

where the value of $\Gamma^\nu_{\alpha\beta}$ can (fairly laboriously) be found to be

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\sigma\mu} \Big(\frac{\partial g_{\alpha\sigma}}{\partial x_\beta} + \frac{\partial g_{\beta\sigma}}{\partial x_\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x_\sigma} \Big) \tag{18}$$

Instead of equation (18), it is often easier just to use equation (16) to calculate the Christoffel symbols.

The set of $\Gamma^\alpha_{\nu\beta}$ are called ***connection symbols*** or ***Christoffel symbols***[15]. They represent terms added to the ordinary partial derivatives in order to correct for the curvature of spacetime, thus allowing us to calculate the value of a tensor field at an infinitesimal distance from the current value. They "connect" the tangent spaces at the two points. They may be viewed as needed to represent the parallel transport of the later vector back to the origin of the first for comparison.

Note that even in flat, Euclidean space where the connection symbols are zero for Cartesian coordinates (x,y,z), this is not true for polar coordinates.

For a one-form, the covariant derivative is different by a sign:

$$\nabla_\beta V_\alpha = \partial_\beta V_\alpha - V_\mu \Gamma^\mu_{\alpha\beta}.$$

One more, worse notation for the covariant derivative is to use a semi-colon:

$$V^\alpha_{\ ;\beta} = V^\alpha_{\ ,\beta} + V^\mu \Gamma^\alpha_{\mu\beta}. \tag{19}$$

Be sure to note the comma and the semi-colon.

For higher rank tensors, to do a covariant derivative of a general tensor with respect to β, ∇_β, take $\partial_\beta T^{\mu\nu\cdots}$ and

for each upper index, V^α, add a μ sum over $\Gamma^\alpha_{\mu\beta}(V^\mu)$

for each lower index, V_α, subtract a μ sum over $\Gamma^\mu_{\alpha\beta} V_\mu$.

So for rank-2 tensors, one gets:.

15 Or sometimes, depending on the circumstances, Levi-Civita symbols. Or, because of their formula, X-awful symbols.

$$\nabla_\beta T^{\mu\nu} = \partial_\beta T^{\mu\nu} + \Gamma^\mu_{\alpha\beta} T^{\alpha\nu} + \Gamma^\nu_{\alpha\beta} T^{\mu\alpha}$$
$$\nabla_\beta T^\mu{}_\nu = \partial_\beta T^\mu{}_\nu + \Gamma^\mu_{\alpha\beta} T^\alpha{}_\nu - \Gamma^\alpha_{\nu\beta} T^\mu{}_\alpha$$
$$\nabla_\beta T_{\mu\nu} = \partial_\beta T_{\mu\nu} - \Gamma^\alpha_{\mu\beta} T_{\alpha\nu} - \Gamma^\alpha_{\nu\beta} T_{\mu\alpha}$$

which will be needed for deriving the Riemann curvature tensor.

In fact, all these results about covariant derivatives and connection symbols can be derived in two-dimensional Euclidean space using polar coordinates, in which the basis vectors depend on position, as in curved space.[16]

16 Schutz, 118-135.

3.5. Parallel transport[17]

In order to derive equations of motion, we must consider movement *along paths* in spacetime. Then we can derive a geodesic, an extremized path equivalent to a Euclidean straight line, the shortest distance between two points. A straight line is also the path which follows its own tangent. What interests us are vector and one-form fields, so we must consider the movement of such objects along a path.

In order to understand tensors in curved spacetime, we can start with vectors in Euclidean space. In order to compare two such vectors, we move one from its own origin to the head or tail of another, keeping it parallel to itself, then add or subtract the two. Such movement of a vector is called ***parallel transport***. This is expressed mathematically by requiring that the vector maintain its length and direction at all points, which can be expressed by

$$\frac{d\vec{v}}{d\lambda} = \frac{d}{d\lambda}\left(v^{\alpha}\hat{e}_{\alpha}\right) = 0 \Rightarrow \frac{dv^{\alpha}}{d\lambda} = 0,$$

where the curve followed by the vector is expressed as a function of a variable λ,

$$\vec{x} = x^{\alpha}(\lambda)e_{\alpha}.$$

In this case, parallel transport works because both points (in fact, all points) are in the same tangent space, the set of all possible vectors at a point in spacetime.

In GR, the curvature of spacetime prevents us from just adding or subtracting the field's values at different places because the spacetime also changes. In another words, the tangent space at each point along the curve is different, so the vector moves from one tangent space to another. This fact is what led us to the covariant derivative.

Consider the 2-dimensional case in figure 1, where the vector is parallel transported around the triangle on the curved 2-d surface from A to B to N, such that it maintains its direction in the tangent plane to the surface at each point along the curve. As it moves around the curve, the tangent plane does not just tilt, it is a new tangent plane, the 2-d equivalent of a tangent space. The tangent plane at each point has a specific orientation. At the end of the circuit, the vector has returned to its original tangent plane, but no longer points in the same direction. This is due to the curvature of the sphere's surface along the path followed by the vector.

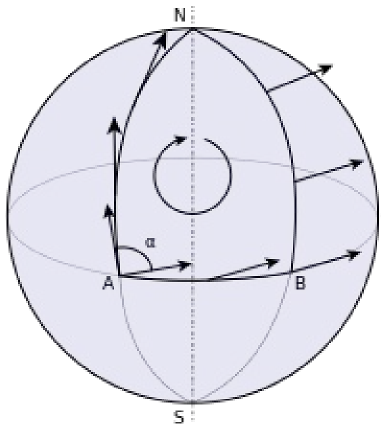

Figure 1: Parallel transport of a vector on a spherical surface, from Wikimedia Commons[18]

This shows clearly that there is no way to compare vectors in two different tangent spaces, so the idea of the relative velocity of, say, two galaxies has no meaning, since their velocity as we see it is not well defined. Expansion of the universe is better seen not as galaxies which are moving farther apart, but

17 I found the notion of parallel transport difficult to grasp. Thanks to the following video (https://www.youtube.com/watch?v=p1tfZD2Bm0w&feature=youtu.be) and to Physics Forum (www.physicsforums.com/threads/understanding-parallel-transfer.911852/#post-5743713 and https://www.physicsforums.com/threads/understanding-parallel-transport.915267/).

18 Parallel transport, https://commons.wikimedia.org/wiki/File:Parallel_Transport.svg.

as change in the metric of the intervening space.

There is a difficulty with Figure (1). We see that the 2-d tangent plane "tilts" in the extrinsic 3-d space in which we imagine it. But we are unable (most of us, at least) of visualizing a 3-d space in a 4-d one somehow "external" to it. We are stuck with the 3-d intrinsic space.

In order to parallel transport a vector in curved space, we will employ the usual trick: We will require two vectors at infinitesimally close points to be parallel and of equal length. We must take into account the curvature of spacetime by using the covariant derivative. By analogy with the Euclidean case, we will define ***parallel transport*** of the vector along the path as requiring that the covariant derivative of the vector along the path vanish. Mathematically, we define the ***directional covariant*** (or ***intrinsic*** or ***absolute***) ***derivative along the path*** as follows:

$$\frac{D}{d\lambda} = \frac{dx^\mu}{d\lambda}\Delta_\mu.$$

For parallel transport of the vector, we require the directional covariant derivative of the vector along its path to be zero, which leads to the ***equation of parallel transport***:

$$\frac{dV^\alpha}{d\lambda} + \Gamma^\alpha_{\nu\beta}V^\nu \frac{dx^\beta}{d\lambda} = 0.$$

Seen slightly differently, the derivative of a vector field along a path parametrized by λ is

$$\frac{d\vec{V}}{d\lambda} = \frac{\partial x^\mu}{\partial\lambda}\frac{\partial\vec{V}}{\partial x^\mu} = t^\mu \nabla_\mu V^\beta \hat{e}_\beta$$

where t^μ is the tangent to the path. So, using the covariant derivative, the condition for parallel transport of a vector becomes

$$t^\mu \nabla_\mu V^\beta = 0 \qquad (20)$$

We need parallel transport because it will enable us to calculate a geodesic and identify the path followed by an object (say, a particle) on a pseudo-Riemannian manifold.

3.6. Geodesics

In Euclidean geometry, the shortest path between two points is a straight line, which is a form of ***geodesic***, the extremized distance between two points. What interests us is that that this is the path followed by a moving particle subject to no external forces. One can state this more precisely: In Euclidean geometry, a particle follows a path in which the tangent to the path at a point is parallel to the tangent at the preceding point. In other words, the particle parallel-transports its own tangent vector, which explains the reason for the preceding section.

We extend this idea into curved space by generalizing the statement that a geodesic is a path which parallel-transports its own tangent vector. Calling the tangent

$$t^\alpha = \frac{\partial x^\alpha}{\partial\lambda}$$

and using parallel transport, we see that a geodesic is represented by the ***covariant directional derivative***

$$\frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = \frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\rho\sigma}t^\rho t^\sigma = 0 \qquad (21)$$

which is therefore called the ***geodesic equation***. In this case, λ is an example of an ***affine*** parameter, a parametrization of a curve “such that the parametric equations for the curve satisfy the geodesic equation.”[19]

19 Planetmath.org, http://planetmath.org/affineparameter.

3.7. The Riemann curvature tensor and friends

The Riemann curvature tensor is defined so as to measure the effect of curvature on a tensor which is parallel transported along different paths or along an infinitesimal loop. It is most easily derived by considering its definition in terms of non-commutating infinitesimal movements. We have seen in Figure 1 how parallel transfer of a vector around a closed path in curved space can lead to a different vector. We can consider an infinitesimal transport along a direction b, then along a direction c and compare this to the same transports in the opposite direction. This may be done by considering the commutator of the two operations:

$$[\nabla_c, \nabla_b]V_a := \nabla_c \nabla_b V_a - \nabla_b \nabla_c V_a \equiv R^d{}_{abc} V_d,$$

which defines the ***Riemann curvature tensor***:

This can be calculated by using the equations at the end of section 3.4 and, taking $\nabla_b V_a$ and $\nabla_c V_a$ to be rank-2 tensors. After some creative fiddling with indexes, one finds

$$R^d{}_{abc} = \partial_b(\Gamma^d_{ac}) - \partial_c(\Gamma^d_{ab}) + \Gamma^e_{ac}\Gamma^d_{eb} +^e_{ab}\Gamma^d_{ec}. \quad (22)$$

It is based on the connection coefficients, which are products of the metric and partial derivatives of the metric.[20] So it's all made up from the metric. The metric knows all.

For a vector,

$$[\nabla_c, \nabla_b]V^a = \nabla_c \nabla_b V^a - \nabla_b \nabla_c V^a = R^a{}_{dbc} V^d$$

and this can be generalized to any type of tensor.

Two other quantities used in GR are derived from the Riemann curvature tensor, the Ricci tensor and the Ricci scalar. The ***Ricci tensor*** is formed by contracting a pair of the Riemann tensor's indexes, either the first and last or the first and third. The result is the same give or take a plus or minus sign. Taking the first and third,

$$R_{\mu\nu} = R^\lambda_{\mu\lambda\nu} \quad (23)$$

The ***Ricci scalar*** is the trace of the Ricci tensor.

$$R = R^\mu{}_\mu = g^{\mu\nu} R_{\mu\nu} \quad (24)$$

20 Collier, 202, says "the Riemann curvature tensor, a glorious mixture of derivatives and products of connection coefficients."

4. Back to physics

So the principle of covariance must take into account curvature and for this, all this math in the form of tensors is needed, because:

> Tensor equations are independent of coordinates an so are the same in every reference frame.

Let's do some physics.

4.1. Four-velocity and four-momentum[21]

The ***four-velocity*** $\vec{U}$ is defined to be a vector tangent to the ***world line*** of a particle, i.e., the path it follows through spacetime, and of length one unit of time in the particle's rest frame. This is equivalent to its time basis vector $\vec{e}_0$. If the particle is accelerated, the four-velocity is defined in an inertial frame which momentarily has the same velocity as the particle, the ***momentarily co-moving reference frame***, or ***MCRF***.[22] In any reference frame, it is the derivative of the components with respect to the proper time τ.

$$U^\mu = \frac{dx^\mu}{d\tau} = \left(\frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}\right) \tag{25}$$

and using the chain rule, we find

$$U^\mu = \frac{dx^\mu}{d\tau} = \frac{dx^\mu}{dt}\frac{dt}{d\tau} = \gamma(c, \vec{v}) \tag{26}$$

Four momentum is simply the particle's four-velocity multiplied by its rest mass:

$$\vec{p} = m\vec{U} = (E, \vec{p}) = (\gamma m, \gamma m\vec{v})$$

where for small velocity

$$E := p^0 = m\gamma \approx m + \frac{1}{2}mv^2 \tag{27}$$

where c=1, as usual. Also,

$$\eta_{\mu\nu}p^\mu p^\nu = -m^2 = -E^2 + p^2$$

so that

$$E^2 = m^2 + p^2 \tag{28}$$

4.2. The energy-momentum tensor

We know that gravity. i.e., the curvature of space, is brought about by energy and mass, which are equivalent in GR. So we must express mass and energy in the form of a tensor. Cosmologists do this by taking advantage of the ***Cosmological Principle***, which states that on very large (humongous) scales, the universe is homogeneous and isotropic. The universe is considered to be a continuum composed of ***elements*** which can be treated as points each one of which possesses unique values for various properties such as energy and momentum. In other words, space is composed of ***fields***, each of which has a value at each point.

As a simplest case, cosmologists define ***dust*** as a collection of particles which are all at rest in some Lorentz frame, their mutually co-moving reference frame. In the MCRF, dust particles have a density of particles, or number density, n, which may vary from point to point. When they are moving relative to an observer, length contraction means that this observer sees a density γn. If the dust is moving with velocity $\vec{v}$, then the flux across a surface of constant $\vec{x}$ is $\gamma n v^{\vec{x}}$. We can thus construct the ***number-***

21 A reminder of what was covered in my SR overview.

22 Schutz (2016), 41.

flux four-vector $\vec{N}$ from the four-momentum $\vec{U}$ as

$$\vec{N} = n\vec{U} = \gamma n(1, \vec{v})$$

In its MCRF, the energy density of dust is the number density times the energy of each particle, which is m in this frame.

$$\rho = nm$$

Seen from a moving frame, both n and m will change by a factor of γ, so that $\rho \to \gamma^2 \rho$, This means the general energy density is not a vector, but a rank-2 tensor.

The ***energy-momentum tensor*** (or ***stress-energy tensor***) is defined such that its $\mu\nu$ component is the flux (rate of flow) of the μth component of four-momentum across a surface of constant x^ν. We can write it in any frame and the simplest is the MCRF where for dust it has only one term:

$$(T^{00})_{MCRF} = \rho = nm$$

which suggests generalization of the tensor to

$$T = \vec{p} \otimes \vec{N}. \tag{29}$$

Then

$$T^{\alpha\beta} = \rho U^\alpha U^\beta, \tag{30}$$

which makes it clear that T is a rank-2 tensor (a vector) and is symmetric, $T^{\alpha\beta} = T^{\beta\alpha}$. Since the energy-momentum tensor component $T^{\alpha\beta}$ is the flux of α momentum across a surface of constant x^β, we see that:

- T^{00} is the energy density;
- T^{0i} is the energy flux across surface x^i, e.g., heat conduction;
- T^{i0} is the momentum density;
- T^{ij} is the flux of momentum I across surface x^j, called the ***stress***.

It is straightforward to write out the components of the tensor using equation (30). For instance:

- $T^{00} = \rho U^0 U^0 = \gamma^2 \rho,$
- $T^{0i} = \rho U^0 U^i = \gamma^2 \rho v^i.$
- $T^{i0} = \rho U^i U^0 = T^{i0} = \gamma^2 \rho v^i,$
- $T^{ij} = \rho U^i U^j = \gamma^2 \rho v^i v^j$

A special case, somewhat more general than dust, is that of a ***perfect fluid*** , which is composed of elements which have no sliding forces (viscosity) between them, only pressure. It amounts to dust plus pressure, without heat conduction or or viscosity in the MCRF. In this case, the energy-momentum tensor in the MCRF, where $U^i = 0$, is diagonal and its (0,0) element has the same value as that of dust, ρ. The spatial diagonal elements are momentum fluxes, i.e., changes in momentum per unit area, and changes in momentum are forces, according to Newton, who is correct in the MCRF. So the spatial diagonal elements all are equal to the pressure, p. In its generalized tensor form, then, the energy-momentum tensor for a perfect fluid is.

$$T^{\mu\nu} = (\rho + p)U^\mu U^\nu + pg^{\mu\nu},$$

since $U^0 = \gamma = 1$ and $U^i = 0$ in the MCRF.

(31)

4.3. Einstein's gravitational field equations

Einstein spent almost ten years figuring out the equations of GR so that they would obey the three principles of covariance, consistency and equivalence. The result can be written as

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa T_{\mu\nu} \tag{32}$$

where

$$\kappa = 8\pi G.$$

Look at the pieces:

- $R_{\mu\nu}$ is the Ricci tensor and R is the Ricci scalar; both are functions of the curvature of space via the connection coefficients and, therefore, of the metric;
- $g_{\mu\nu}$ is the metric and describes the distance between two points, a function of the curvature of space;
- G is Newton's gravitational constant, which is where gravity comes in;
- $T_{\mu\nu}$ is the energy-momentum tensor expressed as a double one-form or tensor of type $\binom{0}{2}$.

So the left-hand side represents the curvature of space expressed as a sum of functions of the metric; the right-hand side, the gravity associated with energy and momentum in space. The equation(s) therefore express the relation between curvature and energy-momentum.

If we include the (in)famous cosmological constant, we get:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \tag{33}$$

which can be rewritten as

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa\Big(T_{\mu\nu} - \frac{\Lambda}{\kappa}g_{\mu\nu}\Big), \tag{34}$$

where it is clear that the cosmological constant may be seen as contributing to the energy-momentum of space through the tensor

$$T^{(\Lambda)}_{\mu\nu} = -\frac{\Lambda}{\kappa}g_{\mu\nu}.$$

In the MCRF, if we assume a perfect-fluid model for $T^{(\Lambda)}_{\mu\nu}$ as in equation (31), then from the matrix elements [ρ_Λ, p_Λ], we can read off the values of the momentum and pressure due to the cosmological constant and find that

$$\rho_\Lambda = \frac{\Lambda}{\kappa} = -p_\Lambda. \tag{35}$$

This shows that a positive density, ρ_Λ, due to the cosmological constant gives a ***negative pressure***, p_Λ. The field equation can be split into parts due to matter (ordinary or dark) and the vacuum (the cosmological constant):

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa\Big(T_{\mu\nu} + T^{(\Lambda)}_{\mu\nu}\Big) = \kappa\Big(T_{\mu\nu} - \rho_\Lambda g_{\mu\nu}\Big). \tag{36}$$

The vacuum (cosmological constant) part of the right-hand side has opposite sign to the energy-momentum tensor, so if the former is the source of gravitational attraction, the latter must be a source of ***repulsion***, driving the universe to ***expand***. This is the basis of the de Sitter model, discussed in paragraph 5.3.3.

Physicists sometimes use the ***Einstein tensor***, defined as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$$

so that equation (32) becomes simpler:

$$G_{\mu\nu} = \kappa T_{\mu\nu}.$$

As we shall see, it is possible to decompose the energy-momentum tensor further, into parts for matter, radiation and the vacuum (or cosmological constant, or dark energy):

4.4. Consistency – the equations of motion

According to the principle of consistency, these equations must reduce to those of Newton in the appropriate limit.

In an inertial frame where no gravity is present, where the connection coefficients are zero, the geodesic equation (21) reduces to

$$\frac{d^2 x^\alpha}{d\tau^2} = 0$$

where the affine parameter λ has been chosen to be the proper time τ. In the non-relativistic limit, τ becomes t, time, and the equation says that the acceleration is zero, consistent with Newton's first law of motion.

The GR equivalent of Newton's second law

$$\vec{F} = m\vec{a} = \frac{d\vec{p}}{dt}$$

is easily shown to be

$$F^\mu = \frac{DP^\mu}{d\tau}$$

and has the same form.

Newton's third law is complicated in GR. Usually, one assumes a large source of gravitation, say a star or a black hole, and considers the way a smaller particle moves along a geodesic in its gravitational field. Turning that around to consider the effect of the particle's tiny mass on the black hole's gravitational field is messy, and according to at least one authority, not addressed by Einstein's GR. So that subject is clearly beyond the scope of this document.

Suffice it here to indicate that in a weak, static gravitational field, the force on a slowly moving particle can be shown to be equal to that of Newton's equation in the form of ***Poisson's equation***

$$\nabla^2 \phi = \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}\right) = 4\pi G\rho$$

where ϕ is the gravitational potential field.

5. What GR tells us

A general solution of the Einstein field equation is impossible. But several solutions have been found based on approximations or limiting cases. These are summarized in the following table.

Model or metric	Simplifying assumptions	Elements	Notes/validity
Schwarzschild	Spherically symmetric, static gravitational field due to single massive spherical body in otherwise empty space	Schwarzschild radius	Stars and black holes
Robertson-Walker	Most general metric conforming to cosmological principle	Scale factor R(t) and curvature k = -1, 0, or 1	Metric of Friedmann equations
Perfect fluid		Dark matter and energy	$\rho_\Lambda > 0$ => $p_\Lambda < 0$ means dark energy
	Plugging R-W metric into Einstein's field equation gives Friedmann equation(s), which lead in turn to several metrics.		
Empty universe ($\rho = 0$)	Empty space	Flat, static Minkowski space (k = 0) or negatively curved expanding universe (k = -1)	
Static Einstein universe	Static solution because of inclusion of cosmological constant	R(t) = constant, $\Lambda > 0$	k = +1, positive curvature, unstable
De Sitter universe	Only dark energy	$\rho_{m,0} = \rho_{r,0} = 0$	Recent universe (>9.8 Gy)
Radiation-only		k = 0, $\rho_{m,0} = \rho_\Lambda = 0$	Early universe (0-50 Ky)
Einstein-de Sitter universe	Mass only (dust)	$\rho_{r,0} = \Lambda = 0$	Mid-term universe; critical density Ω for k = 0

Table 3: Approximate solutions to Einstein gravitational field equations

5.1. The Schwarzschild metric, stars and black holes

The Schwarzschild metric is as follows:

$$ds^2 = -\Big(1 - \frac{2GM}{r}\Big)dt^2 + \Big(1 - \frac{2GM}{r}\Big)^{-1}dr^2 + r^2 d\Omega^2, \tag{37}$$

where

$$d\Omega^2 = d\theta^2 + sin^2\theta d\phi^2 \tag{38}$$

This metric assumes a spherically symmetric, static gravitational field due to a single massive spherical body in a vacuum. The object should not be rotating either, but the solution can still be used for a slowly rotating object, so it is good outside a star like the sun or outside a black hole. It ignores the field due to the mass of the observer, generally a test particle. ***Birkhoff's theorem*** states that this is the only, unique solution for this case. The parameter M is the mass within the radius r and need only be spherically symmetric. G is of course Newton's gravitational constant.

For very large values of r, the metric reduces to the metric for Minkowski space in polar coordinates. This is also true if there is no massive object, so that M=0. In fact, the variables r and t represent real radial distance and clock time only when M=0; otherwise, they are modified by the curvature of spacetime, what we have called coordinate variables (bases).

Just for examples, the Schwarzschild radius R_M measures about 9 mm for a body with the mass of the Earth and 3 Km for that of the Sun, well within the boundary of each body.

The equation (37) indicates a real singularity at r=0, but the seeming singularity at r = 2GM, called the ***Schwarzschild radius***, is a coordinate singularity, due only to the choice of coordinate system. It can be eliminated, e.g., by converting to so-called ***Kruskal coordinates***,

$$u = \left(\frac{r}{2Gm} - 1\right)^{1/2} e^{r/4Gm} cosh(t/4Gm)$$

$$u = \left(\frac{r}{2Gm} - 1\right)^{1/2} e^{r/4Gm} sinh(t/4Gm)$$

for which the Schwarzschild metric becomes

$$ds^2 = \frac{32(Gm)^3}{r} e^{-r/2Gm}(-dv^2 + du^2) + r^2(d\theta^2 + sin^2\theta d\phi^2)$$

and nothing blows up at r=2Gm. It seems like cosmologists spend a lot of time searching for new coordinates which show up better the meaning of equations. Nevertheless, the ***Schwarzschild radius*** is defined as $R_S = 2GM$, remembering that we are using c=1, otherwise, that would be $R_S = 2GM/c^2$.

If the ***proper distance***, $d\sigma$, is taken as the distance with time fixed, then

$$d\sigma^2 = ds^2 = \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2,$$

which for fixed r is

$$d\sigma^2 = ds^2 = r^2 d\Omega^2 = r^2 d\theta^2 + r^2 sin^2\theta d\phi^2,$$

which is just the line element for a sphere in 3-dimensional Euclidean space. So for constant r and t, the Schwarzschild metric defines a sphere in Euclidean space. For fixed angles ($d\theta = d\phi = 0$), we have

$$d\sigma = \left(1 - \frac{2GM}{r}\right)^{-1/2} dr > dr.$$

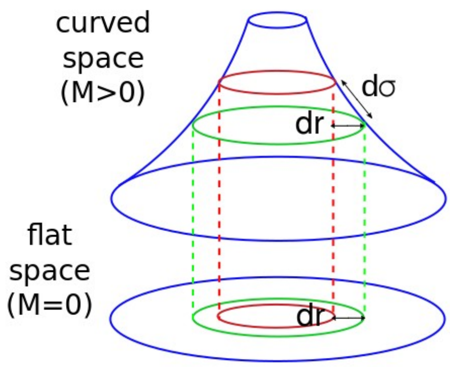

Figure 2: Embedding diagram of radial and proper distance in Schwarzschild spacetime, after Collier

This can be shown somewhat intuitively by an ***embedding diagram***, showing how the difference between $d\sigma$ and dr entails a curved space (Figure (2)).

5.1.1. Gravitational time dilation and redshift

Using the Schwarzschild metric, one can derive the relation between the proper time of an event viewed by a distant observer, say at a nearly infinite distance, and that viewed by an observer at the event.

$$d\tau_\infty = \left(1 - \frac{2GM}{r}\right)^{-1/2} d\tau \tag{39}$$

So to the distant observer, clocks will seem to run more slowly the closer they are to the earth's surface, which may be paraphrased as "gravity makes time run slower." This phenomenon of ***gravitational time dilation*** must be taken into account by GPS systems.

From the same equation (39), the frequency of light emitted by the owner of the clock in question will be seen by the distant observer as lower by

$$f_\infty = \left(1 - \frac{2GM}{r}\right)^{1/2} f_{em}$$

and so shifted towards the red (at least, if the emission frequency is higher than that of red, which is the case for most visible light). This is the ***gravitational redshift*** and has nothing to do with that of SR or a whatsoever velocity-related Doppler effect.

The geodesic equation of the Schwarzschild metric is messy to calculate. But once known, it can be used to predict both the the precession of the perihelion of Mercury, the experimental confirmation of which made Einstein world-famous, and the deflection of light in a gravitational field, leading to the phenomenon of ***gravitational lensing.***

5.1.2. A bit of history

This is about as good a time as any for a bit of history.

In 1905, Einstein published the special theory of relativity (SR). In 1907, he started thinking about free fall and acceleration, finally presenting his theory of general relativity to the Prussian Academy of Science in November 1915.

Karl Schwarzschild quickly found the metric named after him in December 1915, but unfortunately died of an autoimmune disease in 1916 while serving on the Russian front.

In 1939, Robert Oppenheimer and one of his students, Hartland Snyder, used relativity to show that someone falling into a black hole was never seen by a distant observer to arrive at the event horizon, although the observer himself definitely did get there in a finite (and for her all-too-short) period of time.

The term "black hole" was first used by theoretical physicist John Wheeler in 1967.

5.1.3. Schwarzschild black holes

According to NASA, "[b]lack holes are really just the evolutionary end points of massive stars."[23] It is an understatement to say that black holes have a number of interesting properties.

The gravitational field of a black hole is so strong that anything – even light – which falls into it, i.e., which gets within a distance of the Schwarzschild radius of the central singularity, will be drawn toward the singularity without any possibility of escape, as to do so would require a velocitiy greater than the speed of light. So the Schwarzschild radius describes a surface of no return, the ***event horizon***. Although the existence of the event horizon is a defining characteristic of black holes, it is not a rigid surface. Outside the event horizon, one could pass close to the black hole and then move away – if one calculates his orbit correctly. The black hole is not a cosmic vacuum cleaner, sucking up

23 NASA chat. "Why do black holes suck? Or do they?" https://www.nasa.gov/connect/chat/black_hole_chat.html.

everything around it. But anything or anyone penetrating the event horizon is indeed doomed.

Since the black hole's gravity keeps in everything which penetrates the event horizon, including light, the black hole is … black – and so invisible to observers. As light approaches it, a distant observer sees it as being increasingly gravitationally red-shifted until it also becomes black and disappears from view.

Interestingly, already near the end of the 18th century, John Michell and, slightly later, Pierre-Simon Laplace, conjectured that light, considered at the time to be composed of particles, would be attracted by gravity and so could only escape from a a planet's gravitational field if its speed, c, exceeded the escape velocity given by

$$c = \sqrt{\frac{2GM}{R}}$$

meaning that it could not escape if it were at a radius

$$R < \frac{2GM}{c^2}$$

which is the Schwarzschild radius. Neither of these men knew anything about black holes, but they theorized that very heavy stars would be dark because light could not escape from them. Laplace surmised that the largest objects in the universe might be invisible to us.

The gradient of the gravitational field within the event horizon is so strong that any poor astronaut who had the mishap of falling feet first "into" the hole, i.e., penetrating the event horizon, soon would feel a much stronger force on her feet than on her head so that, instants before before being torn apart, she would be stretched out, what cosmologists refer to jocularly as "spaghettification". Such forces, due to the non-uniformity a gravitational field are called ***tidal forces***.

It is possible to find approximate solutions to the equations both for distant observers and for the poor in-falling astronaut. Strangely enough, the distant observer of the doomed astronaut approaching the event horizon would "see" the astronaut's time slow down due to gravitational time dilation. Every incremental distance of the falling astronaut would last longer, so that the observer would in fact never see the her fall in, i.e. disappear. This is also true if it is not an astronaut falling in, but the surface of a compacting star in the process of forming a black hole. One cannot see a black hole form because there is not enough time.

Incidentally, this is a good way to travel forward in time: Lurk for an hour or so close to a black hole event horizon, then go home to find centuries have gone by.

The unfortunate astronaut's clock does not slow down, though, so she does indeed fall into the black hole, as does the surface of a collapsing star.

Stranger yet, the Schwarzschild metric (37) shows clearly that inside the event horizon, the dt term becomes positive and the dr term, negative. In some sense, the coordinates of time and radial distance have been exchanged so that falling in distance has become falling in time and vice versa. *Comprenne qui pourra*.[24]

Cosmologists say that "black holes have no hair", known as the ***no-hair theorem***. This statement refers to the fact that black holes have only three properties which can be measured by an external observer: mass, angular momentum and electric charge. Cosmologists have found several metrics to describe black holes with combinations of these properties:

There are two classes of black holes in the visible universe:

- huge ***quasars*** and ***active galactic nuclei*** (or ***AGN***, possessing a compact, luminous core) whose masses are between 10^6 and 10^9 solar masses, and
- ***microquasars*** and stellar-mass black holes, whose masses are on the order of 10 solar masses.

Note that these classes are based on mass, not size. The energy observed is outside the event

24 Let he who can understand do so.

horizon, as explained below.

Quasars are thought to start out relatively small and grow by absorbing nearby stars. Active black holes are the most powerful continuous energy sources in the universe. It is thought that there Is a quasar-class black hole in the center of each galaxy, including our own, the Milky Way. Stars near the center are moving so fast that their motion can only be explained as being due to a black hole.[25]

A non-rotating black hole is spherical, but a rotating one will bulge out at its equator. The Kerr metric for a rotating black hole defines a region called the ***ergosphere*** where spacetime is ***dragged*** in the direction of rotation at a speed greater than the speed of light relative to the rest of the universe. This is due to an effect called the ***gravitomagnetism***. It is so named by analogy with the Lorentz force of electromagnetism because in the presence of such a field, the gravitational acceleration of a particle depends on its velocity.[26]

Metric	Properties taken into account
Schwarzschild	Only mass
Kerr	Mass and angular momentum
Reissner-Nordström	Mass and electric charge
Kerr-Newmann	Mass, angular momentum and electric charge

Table 4: Black-hole metrics

As matter possessing angular momentum is attracted toward a black hole, it goes off to the side a bit and follows a path around the hole, Such matter forms a flat disk around the black hole, an ***accretion disk***. Somehow, such matter loses angular momentum and spirals into the black hole, releasing its energy as heat and radiation, mainly X-rays. The radiation follows paths curved by the black hole's gravity. Observation of this radiation has identified many possible black holes.

Figure 3: Jets from super-massive black hole in the center of the 4C+29.30 galaxy, from NASA.[27]

Streams of matter also emerge in narrow ***jets*** with opening angles of $<5°$, thought to be emitted perpendicularly to the plane of the accretion disk and in both directions. Jets may be due to magnetic fields produced by the rotating disk and are an important subject of black-hole studies.[28]

It is predicted that black holes also possess a finite temperature and entropy. If this were true, black

25 NASA chat. "Why do black holes suck? Or do they?" https://www.nasa.gov/connect/chat/black_hole_chat.html

26 Schuts (2003), 245.

27 Black-hole powered jets plow into galaxy. https://www.nasa.gov/multimedia/imagegallery/image_feature_2510.html

28 For more on accretion disks and jets, see Abramowicz and Fragile, "Foundations of black hole accretion disk theory", online at https://arxiv.org/pdf/1104.5499.pdf.

holes would not be completely black. QM indeterminacy would form matter-antimatter pairs near the surface, using energy from the black hole itself. If one member of such a pair is closer and falls into the black hole, the other may escape, carrying off some of the energy. This is called ***Hawking radiation***, after its discoverer. If this happens, then black holes may eventually decrease in mass until they “fade” away over a very long period of time, the rate of Hawking radiation being tiny.[29] The effects of entropy and information in the case of black holes are widely debated, partly because there is no way to test the theories, the singularity in the center of the black hole being unattainable.[30]

5.2. The cosmological principle

Cosmologists have taken advantage of the cosmological principle to look for a metric describing a spacetime which is homogeneous and isotropic.

5.2.1. Robertson-Walker metric

The most fruitful method so far has been to idealize the case as follows. Spacetime evolves in time so that at each value of time it consists of a hypersurface which is homogeneous and isotropic. This assumption leads to the ***Robertson-Walker metric***, the most general possible metric which describes such a spacetime. It can be written as follows:

$$ds^2 = -dt^2 + R^2(t)\Big[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\Big] \quad (40)$$

where

$$d\Omega^2 = d\theta^2 + sin^2\theta d\phi^2$$

The coordinates of this metric are rather special. The time coordinate t represents the ***cosmic time***, i.e., time measured by an observer whose motion is only due to the expansion of space, called a ***fundamental observer***, and whose ***peculiar motion*** is zero. This would be the time seen by a galaxy, for example. The spatial coordinates, called ***co-moving coordinates***, are measured by a fundamental observer and are constant with time, all time dependence being found in the ***scale factor***, R(t), which shows how big the space part is as it increases (decreases) with time for an expanding (contracting) universe. To repeat, the spatial coordinates do not change with time: Each galaxy has a fixed set of spatial coordinates which lead to a temporally increasing metric only because of the scale factor. The parameter k is the ***curvature*** and is considered (i.e.,normalized) to take on the values -1, 0 or +1. Any other integer can be reduced to one of these three by judicious normalization of the coordinate r,

The three permitted values of curvature lead to different types of universes:

- In the case of k=0, this reduces to ***flat*** Euclidean space.
- For k=+1. It can be reduced to the metric for the surface of a three-sphere and is called the ***closed***, or ***spherical*** R-W metric.
- For k=-1, it gives a hyperbolic, or ***open***, R-W metric.

5.2.2. The expanding universe

Consider only the radial part of the distance at a fixed time,

$$d\sigma = R(t)\Big[\frac{dr^2}{1-kr^2}\Big]^{1/2}.$$

This can be integrated for the three cases and, after some manipulation, all three give the same result:

29 Can you imagine something black fading?

30 See, for instance, Sabine Hossenfelder, “Why do physicists worry so much about the black hole information paradox?”. http://backreaction.blogspot.fr/2017/04/dear-dr-b-why-do-physicist-worry-so.html

$$\frac{d\sigma}{dt} = \Big(\frac{1}{R}\frac{dR}{dt}\Big)\sigma := H(t)\sigma \qquad (41)$$

where H(t) is the ***Hubble parameter***. Since the time derivative is a radial velocity, we get for t=0 (i.e., now)

$$v = H_0 d$$

which is ***Hubble's law***, telling us that the farther distant (d) a star, the faster it is receding from us – and from everything else. Among other things, this says that galaxies or stars farther than $d_H = c/H_0$ are receding from us faster than the speed of light. Not to worry, this is because of space expanding, nothing is traveling at superliminal speed in anybody's Lorentz rest frame.

5.2.3. The Friedmann equations

Using a metric to calculate connection coefficients and then pushing all that into the various tensors is tedious, to say the least. So many cosmologists start with a predigested version of those results. The R-W metric can be used to calculate the connection coefficients, then the Riemann curvature tensor and the quantities derived from it, the Ricci tensor and scalar. Putting these and the perfect-fluid energy-momentum tensor into the Einstein equation (32) leads eventually to the Friedmann equations. The first, called the ***Friedmann equation***, is

$$\Big[\frac{1}{R}\frac{dR}{dt}\Big]^2 = \frac{8\pi G}{3}\rho - \frac{k}{R^2} \qquad (42)$$

and the second, the ***Friedmann acceleration equation***, is:

$$\frac{1}{R}\frac{d^2R}{dt^2} = -\frac{4\pi G}{3}\big(\rho + 3p\big) \qquad (43)$$

The sum $(\rho + 3p)$ on the right-hand side of this equation is referred to as the ***active gravitational mass***.[31]

Breaking down the energy density ρ into its three components, we can determine their evolution as following this equation

$$\rho(t) = \rho_{m,0}\Big(\frac{R_0}{R(t)}\Big)^3 + \rho_{r,0}\Big(\frac{R_0}{R(t)}\Big)^4 + \rho^{\Lambda}. \qquad (44)$$

So, defining a ***normalized scale factor***

$$a(t) = \frac{R(t)}{R_0} \qquad (45)$$

allows the Friedmann equation to be written in another much-seen form:

$$\Big[\frac{1}{a}\frac{da}{dt}\Big]^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \qquad (46)$$

or

$$\Big(\frac{\dot{a}}{a}\Big)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}. \qquad (47)$$

Similarly, if we assume pressure follows an equation of statement

$$p = \omega\rho$$

(w = 0, 1/3 or -1 in the case of dust, radiation or dark energy, respectively), then we can find the evolution of the pressure with time

31 Schutz (2003), 242. Schutz says it is responsible for the gravitational redshift and the gravitoelectric field, that "part of the gravitational field that is most like the Newtonian gravitational acceleration.

$$p(t) = \frac{\rho_{r,0}}{3}\Big(\frac{R_0}{R(t)}\Big)^4 - \rho_\Lambda \tag{48}$$

5.3. Other models of the universe

The following models of the universe are all calculated using the Friedmann equations and therefore are based on the R-W metric and the energy-momentum tensor for a perfect-fluid.

5.3.1. Empty-universe model

The simplest case is a completely empty universe:

$$\rho_m = \rho_r = \rho_\Lambda = 0$$

in which case the Friedmann equation becomes

$$\Big[\frac{1}{R}\frac{dR}{dt}\Big]^2 = -\frac{k}{R^2}$$

or

$$\frac{dR}{dt} = \sqrt{-k},$$

so k must be equal to 0 or -1. If k=0, the result is R constant and we have a very boring, empty, unchanging space. Taking k=-1, integrating and dropping the constant factor gives $R = \pm t$ ($R = \pm ct$ if $c \neq 1$). In terms of the above normalized scale factor,

$$a(t) = \frac{t}{t_0}.$$

So an empty universe either remains static and empty or increases linearly with time.

5.3.2. Static Einstein model

Einstein did not like the idea of an expanding space and so finagled his equation to give a static one, adding in the cosmological constant, Λ. Beginning with that version of the Einstein equation (33), the Friedmann equations can be rederived to give

$$\Big[\frac{1}{R}\frac{dR}{dt}\Big]^2 = \frac{8\pi G}{3}\rho - \frac{k}{R^2} + \frac{\Lambda}{3}$$

and

$$\frac{1}{R}\frac{d^2R}{dt^2} = -\frac{4\pi G}{3}\Big(\rho + 3p\Big) + \frac{\Lambda}{3}.$$

In these equations, the vacuum energy is included in the Λ term, so $\rho = \rho_m + \rho_r$ only. The universe can only be static if the pressure p=0 and

$$\frac{dR}{dt} = \frac{d^2R}{dt^2} = 0.$$

The second equation above then says that

$$0 = -\frac{4\pi G}{3}\rho + \frac{\Lambda}{3},$$

so

$$4\pi G\rho = \Lambda$$

in agreement with equation (35). Adding these conditions to the first Friedmann equation above,

$$0 = \frac{8\pi G}{3}\rho - \frac{k}{R^2} + \frac{\Lambda}{3} = \frac{8\pi G}{3}\rho - \frac{k}{R^2} + \frac{4\pi G\rho}{3},$$

or

$$4\pi G\rho = \frac{k}{R^2}$$

so k must be positive and is taken to be k=+1 and the curvature of the static universe is positive. The problem with this solution, as Einstein realized, is that it is extremely unstable and therefore unlikely to be true.

Incidentally, equation (35) gives the equivalence between Λ and ρ_Λ, from which the first equation here may be written as

$$\left[\frac{1}{R}\frac{dR}{dt}\right]^2 = \frac{8\pi G}{3}(\rho + \rho_\Lambda) - \frac{k}{R^2}$$

which is equivalent to the original Friedmann equation, now including ρ_Λ instead of Λ itself.. This is how the cosmological constant leads to a static universe, by providing the effective energy density needed to combat expansion.

So it seems clear that neither of these two examples, the empty universe or the static one, are very useful descriptions of our universe.

5.3.3. De Sitter model

The de Sitter universe goes a step beyond the empty universe by adding vacuum energy only, leaving $\rho_{m,0} = \rho_{r,0} = 0$ as well as k=0. The Friedmann equation (42) then gives

$$\frac{dR}{dt} = \sqrt{\frac{8\pi G\rho_\Lambda}{3}}R$$

which can be resolved somewhat laboriously to give

$$R(t) = R_0 e^{\sqrt{\frac{8\pi G\rho_\Lambda}{3}}(t-t_0)}.$$

In this case the Hubble parameter is

$$H(t) = \frac{1}{R}\frac{dR}{dt} = \sqrt{\frac{8\pi G\rho_\Lambda}{3}}$$

so that

$$R(t) = R_0 e^{H_0(t-t_0)}.$$

Because of the small size of the Hubble constant (2.27x10^{-18}s^{-1}), this exponential curve is barely distinguishable from the t axis, but the universe is inexorably expanding. The expansion is entirely due to the positive cosmological constant.

5.3.4. Radiation-only model

As its name implies, this model assumes the initial matter energy density and the vacuum energy are zero, as well as k: $k = 0 = \rho_\Lambda = \rho_{m,0}$. The Friedmann equation (42) and (44) gives

$$\frac{dR}{dt} = \sqrt{\frac{8\pi G}{3}\rho_{r,0}\frac{R_0^2}{R}} \qquad (49)$$

This can be solved to give

$$R(t) = R_0\left(\frac{t}{t_0}\right)^{1/2}$$

or equivalently

$$a(t) = \left(\frac{t}{t_0}\right)^{1/2}.$$

5.3.5. Einstein-de Sitter model

This model assumes $k = 0 = \rho_\Lambda = \rho_{r,0}$, in other words, a universe of nothing but matter, assumed to behave like dust. So, one more time, the Friedmann equation (42) gives:

$$\frac{dR}{dt} = \sqrt{\frac{8\pi G}{3}\rho_{m,0}\frac{R_0^{3/2}}{R^{1/2}}},$$

which describes an increasing scale factor, R(t), whose rate of increase decreases in time. After some time and labor, the equation renders

$$R(t) = R_0\left(\frac{t}{t_0}\right)^{2/3}$$

or equivalently

$$a(t) = \left(\frac{t}{t_0}\right)^{2/3}.$$

So this model describes a universe which expands forever, but at a continually decreasing rate, as seen from equation (49). This was considered the best model until the late 1990s and the discovery that the expansion is increasing.

5.3.6. Different contributions to density and pressure

We can take the method of equation (36) farther and decompose the the energy-momentum tensor into three parts due to three different energy densities, for matter, radiation and vacuum energy (the cosmological constant):

$$\rho(t) = \rho_m(t) + \rho_r(t) + \rho_\Lambda \tag{50}$$

where the last term indicates that the vacuum energy density is a constant across spacetime. As the energy scale factor increases, the matter and radiation energy densities decrease by a volume factor of R^3. In addition, since the energy of radiation is inversely proportional to its wavelength, which will increase with R, its density goes down by a further factor of R. One can show that the total energy density evolves as[32]

$$\rho(t) = \rho_{m,0}\left(\frac{R_0}{R(t)}\right)^3 + \rho_{r,0}\left(\frac{R_0}{R(t)}\right)^4 + \rho_\Lambda \tag{51}$$

and the pressure evolves as

$$p(t) = \frac{1}{3}\rho_{r,0}\left(\frac{R_0}{R(t)}\right)^4 - \rho_\Lambda,$$

where we assume that matter is in the form of dust and we give or take a few factors of c (=1).

These two equation show how the three forms of energy density and pressure vary with the expansion of the universe. Radiation energy density decreases faster than matter energy density, but both decrease faster than the vacuum energy density, which does not decrease at all. So consideration of these relative energy densities suggests that the universe has been through three periods[33]:

1. an initial, short period of radiation dominance, estimated to measure on the order of 50,000 years, approximated by the radiation-only model;

32 See Collier, 300.

33 Figures taken from Collier, 316.

2. a longer period of matter dominance, estimated as lasting about 9.8×10^9 years, best represented by the Einstein-de Sitter model (EdS);
3. a long period of dominance of vacuum energy, still running, approximated by the de Sitter model.

Equation (48) implies that similar considerations hold for radiation and vacuum pressure, with radiation pressure decreasing rapidly from its initial value.

5.4. More on densities

The Friedmann equation can be written in terms of the Hubble parameter defined in equation (41) as follows:

$$\left(\frac{1}{R}\frac{dR}{dt}\right)^2 = H^2(t) = \frac{8\pi G}{3}\rho - \frac{k}{R^2} \tag{52}$$

where ρ represents the sum of all three energy densities. There are three cases to consider according to the values of k.

1. If the space is flat, then k=0 and
$$\rho = \frac{3H^2(t)}{8\pi G} := \rho_c$$
where ρ_c is the ***critical density*** necessary for a flat space.
2. If $\rho < \rho_c$, then equation (52) shows that k must be negative and therefore k=-1, so the curvature of space is negative. It also shows that R(t) will go on increasing forever.
3. If $\rho > \rho_c$, then k=1. In the context of the Einstein-de Sitter model, where $\rho = \rho_m$, we have seen that ρ_m, as a density, goes down as R^3, so that eventually $H^2(t)$ will become zero and the universe will collapse.[34]

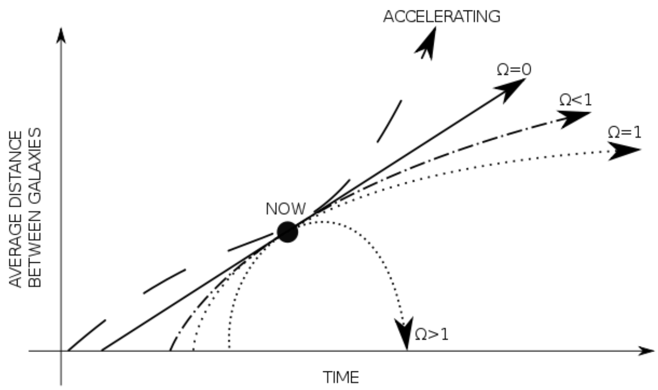

Figure 4: Effect of densities on expansion of space, by Durand via Wikimedia Commons[35].

Cosmologists also employ another set of ***density parameters*** based on the ratio of each energy density to the critical density

$$\Omega_x = \frac{\rho_x(t)}{\rho_c(t)}$$

for x = m, r or λ. The total density is the sum of the three

$$\Omega(t) = \Omega_m(t) + \Omega_r(t) + \Omega_\Lambda(t). \tag{53}$$

Inserted into the Friedmann equation, this gives

34 Whatever does imaginary H(t) mean?

35 https://commons.wikimedia.org/wiki/File:Universe.svg

$$\frac{k}{H^2(t)R^2(t)} = \Omega(t) - 1.$$

So we have the following correspondences:

$$k = -1 \Rightarrow \Omega(t) < 1$$

$$k = 0 \Rightarrow \Omega(t) = 1$$

$$k = +1 \Rightarrow \Omega(t) > 1.$$

If one assumes no radiation and $\Omega_{m,0} + \Omega_{\Lambda,0} = \Omega_0 = 1$ (k=1), one can use the Friedmann equation to calculate the age of the universe to be 13.78 Gy, which is damn close to the WMAP determination of 13.7 ± 0.13 Gy.[36]

5.5. *Gravitational waves*

Beyond this document. They should exist but be very weak. LIGO and VIRGO have found them.

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7. Annex: All you need to solve Einstein's equations

Start with the equation itself, cosmological constant and all, equation (33):

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}.$$

Next, we need the Ricci tensor and scalar, from equations (23) and (24):

$$R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu}$$

$$R = R^{\mu}{}_{\mu} = g^{\mu\nu}R_{\mu\nu}$$

and that requires the Riemann curvature tensor's, from equation (22):

$$R^{d}{}_{abc} = \partial_b(\Gamma^d_{ac}) - \partial_c(\Gamma^d_{ab}) + \Gamma^e_{ac}\Gamma^d_{eb} +^{e}_{ab}\Gamma^d_{ec}$$

We also need the Christoffel or connection symbols, from equation (18):

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2}g^{\sigma\mu}\Big(\frac{\partial g_{\alpha\sigma}}{\partial x_\beta} + \frac{\partial g_{\beta\sigma}}{\partial x_\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x_\sigma}\Big)$$

Now all you need is a metric, g, and an energy-momentum tensor (make it easy on yourself and use the perfect-fluid version) and you're ready to go!

36 Collier, 316-7 and WMAP, https://map.gsfc.nasa.gov/resources/edactivity1.html.