Principles of quantum mechanics

After Leonard Susskind, Quantum mechanics, the theoretical minimum.

1. Spaces and vectors

The space of states of a quantum-mechanical system is a mathematical ***vector space***, an abstract construction of a type of spaces called ***Hilbert spaces***.

A state of the system is a represented by a unit (normalized) vector in the (vector) space of states. A physical observable is described by a linear operator.

Using Dirac's bra-ket notation, a vector state is a ***ket*** represented by $|A\rangle$. Some properties:

$|A\rangle$ +$|B\rangle$ = $|C\rangle$ = $|B\rangle$ + $|A\rangle$

{$|A\rangle$ + $|B\rangle$} + $|C\rangle$ = $|A\rangle$ + {$|B\rangle$ + $|C\rangle$}

There exists a zero: $|A\rangle$ + **0** = $|A\rangle$

For each ket $|A\rangle$ there is a unique $|-A\rangle$: $|A\rangle$ + $|-A\rangle$ = 0.

Kets are linear.

2. Observables, eigenvectors and eigenfunctions

Several principles govern observables and basis vectors.

(1) A physical observable is represented by a linear, Hermitian operator $\mathbf{L}$.

$$\mathbf{L}|A\rangle = |B\rangle$$

The ***Hermitian conjugate*** of an operator is formed by interchanging rows and columns in its matrix representation (operation T) and by taking the complex conjugate of each (*).

$$\mathbf{M}^\dagger = [\mathbf{M}^{\mathrm{T}}]^*$$

(2) Observables, the possible results of measurements, are eigenvalues of the Hermitian operator representing the observable

$$\mathbf{L}|\lambda >= \lambda|\lambda\rangle$$

where λ is the ***eigenvalue*** of the eigenvector ket |λ>. A ***Hermitian operator*** is one which is equal to its Hermitian conjugate

$$\mathbf{M}^\dagger = \mathbf{M},$$

which garantees that its eigenvales are positive and real.

(3a) Unambiguously distinguishable states are represented by orthogonal vectors. i.e., for two eigenvalues, $\lambda_1 \neq \lambda_2$, the corresponding eigenvectors are orthogonal:

$$\langle\lambda_1\lambda_2\rangle = \langle\lambda_2|\lambda_1\rangle = 0 \qquad (1)$$

(3b) The eigenvectors of an operator are a complete set. This means that any operator can be expressed as

$$|V\rangle = \sum_i \alpha_i |V_i\rangle \qquad (2)$$

In other words, “[t]he eigenvectors of a Hermitian operator form an orthonormal basis.”

3c) Two orthogonal eigenvectors may have the same eigenvalue, in which case they are referred to as

degenerate states.

(4) For a state vector $|A\rangle$ and observable $\mathbf{L}$, the probability of observing value λ_i by measurement is

$$P(\lambda_i) = \langle A|\lambda_i\rangle\langle\lambda_i|A\rangle \quad (3)$$

when

$$\mathbf{L}|\lambda_i\rangle = \lambda_i|\lambda_i\rangle.$$

3. Evolution in time

Susskind calls the minus-first law that information is never lost. This means that distinctions are conserved, the QM version of which is ***unitarity***, the fifth prinicple.

(5) The evolution of state-vectors with time is ***unitary***.

This means that distinctions are conserved, which in turn means that eigenvectors are orthogonal. If

$$\langle\Psi(0)|\Phi(0)\rangle = 0 \quad (4)$$

and the time-development operator **U** is defined by

$$|\Psi(t)\rangle = U(t)|\Psi(0)\rangle \quad (5)$$

then orthogonality leads to U being unitary, so that

$$\mathbf{U}^\dagger\mathbf{U} = \mathbf{I} \quad (6)$$

is the identity, i.e.

$$\langle i|\mathbf{U}^\dagger(t)\mathbf{U}(t)|j\rangle = \delta_{ij}. \quad (7)$$

For small time ϵ, we can write

$$\mathbf{U}(t) = \mathbf{I} - \epsilon\mathbf{H} \quad (8)$$

which leads to

$$\mathbf{H}^\dagger = \mathbf{H} \quad (9)$$

and then (tossing in $\hbar$ to correct units)

$$i\hbar\frac{\partial|\Psi\rangle}{\partial t} = \mathbf{H}|\Psi\rangle \quad (10)$$

which is the ***generalized*** (***time-dependent***) ***Schrödinger equation***. For another operator $\mathbf{L}$, one can show

$$\frac{d\langle L\rangle}{dt} = -\frac{i}{\hbar}\langle[L,H]\rangle \quad (11)$$

where the ***commutator***

$$[L,M] = LM - ML \quad (12)$$

Is in general not equal to zero. In classical mechanics, the time derivative can be expressed in terms of Poisson brackets as

$$\dot{F} = \{F,H\}$$

so that the correspondence between classical and quauntum mechanics is

$$[F,G] \Leftrightarrow i\hbar\{F,G\} \quad (13)$$

where the smallness of $\hbar$ makes the term negligible in the classical limit. Then equation (11) in classical mechanics becomes

$$\frac{dL}{dt} = \{L,H\} \quad (14)$$

So if the commutator of an observable with the Hamiltonian is zero, the quantity is conserved. The simplest case

$$[H, H] = 0$$

assures that energy is conserved. Possible energy states are represented by the eigenvalues of the Hamiltonian:

$$\mathbf{H}|E_j\rangle = E_j|E_j\rangle \tag{15}$$

Since the eigenvectors form a complete set of basis vectors,

$$|\Psi\rangle = \sum_i \alpha_i |E_i\rangle \tag{16}$$

and if the Hamiltonian does not depend explicitly on time (p 123), then

$$\alpha_j(t) = \alpha_j(0) e^{-\frac{i}{\hbar}E_j t}. \tag{17}$$

Since $\alpha_i(0) = \langle E_i|\Psi(0)\rangle$, then

$$|\Psi(t)\rangle = \sum_i |E_i\rangle\langle E_i|\Psi(0)\rangle e^{-\frac{i}{\hbar}E_j t} \tag{18}$$

4. Simultaneous observables and uncertainty

It is clear from

$$\mathbf{LM}|\lambda, \mu\rangle = \lambda\mu|\lambda, \mu\rangle$$

that if $|\lambda, \mu\rangle$ is a simultataneous eigenvector basis for the two operators $\mathbf{L}$ and $\mathbf{M}$, then the result is independent of order and the two operators must commute:

$$[\mathbf{L}, \mathbf{M}]|\lambda, \mu\rangle = 0.$$

The case of non-commutation plus some geometry leads to the generalized uncertainty principle:

$$\Delta A \Delta B \geq \frac{1}{2}\langle\Psi|[A, B]|\Psi\rangle \tag{19}$$

5. Composite systems and entanglement

A state vector for two different, independent measurements or subsystems can be represented as a product of the two. But this is only one example of a state vector for a composite system, In general, they are not product states. For a product state, there exist two sets of normalization equations. For the case of two states each with two eigenvalues, each is represented by two complex components and so four variables, but normalization reduces each by one and a negligible phase factor eliminates another, leaving only four independent components (degrees of freedom) for the product state. For a state vector for the same subsystems which is not a product state, there is only one normalization requirement and one phase factor, so there remain six independent parameters, making this a more complicated state. Such a state Is said to be ***entangled***.

The state of a system is a sum over basis vectors labelled by a collection of eigenvalues of mutually commuting operators. For instance, a may represent the eigenvalues of operator A, b, of operator B and so on, so that the state vector for the composite system is

$$|\Psi\rangle = \sum_{ab\ldots}\langle ab\ldots|\Psi\rangle|ab\ldots\rangle = \sum_{ab\ldots}\psi(ab\ldots)|ab\rangle$$

where the set of components or projections

$$\psi(ab\ldots) = \langle ab\ldots|\Psi\rangle$$

is called the set of ***wave functions*** for the system.

For a single state described by a state vector $|\Psi\rangle$, the basic formula for the ***expectation value*** of an observable whose operator is L is

$$\langle L\rangle = \langle\Psi|L|\Psi\rangle = \sum_{a\prime,a}\langle a\prime|\psi^*(a\prime)L\psi(a)|a\rangle = \sum_{a\prime,a}\psi^*(a\prime)\psi(a)L_{a\prime a}$$

where

$$L_{a\prime a} = \langle a\prime|L|a\rangle$$

is the matrix element of the operator L and a represents a complete set of observables for the system.[1]

A composite system of two or more subsystems uses basis vectors labelled by eigenvalues of commuting operators for both subsystems. Such a system may or may not be described as a product system. Consider the case of two systems, Alice and Bob, with respective observables a and b, The composite state is formed by the tensor product of the subsystem state vectors and the composite wave functions are

$$\psi(a,b) = \langle ab|\psi\rangle.$$

Suppose Alice's and Bob's subsystems have observables represented by the operators L or M, each of which operates only on its own subsystem and leaves the other unchanged. Then

$$L_{a\prime a,b\prime b} = \langle a\prime b\prime|L|ab\rangle = \delta_{b\prime b}L_{a\prime a} \qquad (20)$$

and similarly for Bob's subsystem.

if the two systems are completely independent (pure), the composite wave function may be given by a product of two wave functions. In this case.

$$\psi(ab) = \psi_A(a)\phi_B(b).$$

For such a product state,

$$\langle LM\rangle = \langle L\rangle\langle M\rangle$$

but for an entangled state, the ***correlation***

$$C(L,M) = \langle LM\rangle - \langle L\rangle\langle M\rangle \neq 0,$$

making this one way to test for an entangled system.

As an example, cuppose that Alice and Bob each measure a two-valued observable (a spin, up or down) represented by an operator (L or M). For the singlet state,

$$|sing\rangle = \frac{1}{\sqrt{2}}(|ud\rangle - |du\rangle),$$
$$\langle\sigma_{zA}\rangle = \langle\sigma_{zB}\rangle = 0$$

but

$$\langle\sigma_{zA}\sigma_{zB}\rangle = -1.$$

This means nothing is known about what a measurement of σ_{zA} or σ_{zB} will give, but once one is measured, the other is known. This state is ***maximally entangled***. The singlet state is an eigenstate of the operator

$$\vec{\sigma}_A\cdot\vec{\sigma}_B = \sigma_{xA}\sigma_{xB} + \sigma_{zA}\sigma_{zB} + \sigma_{zA}\sigma_{zB}$$

with eigenvalue -1, although this observer is totally unknown to the separate subsystems. The full state is maximally characterized but *nothing* is known about the individual subsystems.

It is not possible to isolate the wave functions of one subsystem of an entangled system. If we want to find the values of observables for one subsystem, the best we can do is use the density matrix, which describes the expected values. Since the observable L has no effect on the subsystem b part of the state, as in equation (20), the expected value of L is

1 This is a VIE, a Very Important Equation.

$$\langle L\rangle = \sum_{a\prime,a,b\prime,b} \psi^*_{a\prime.b\prime} L_{a\prime a}\delta_{b\prime b}\psi_{ab} = \sum_{a\prime,a} \rho_{aa\prime} L_{a\prime a} \qquad (21)$$

where the ***density matrix***

$$\rho_{aa\prime} = \sum_b \psi^*_{a\prime b}\psi_{ab} \qquad (22)$$

represents all A neds to know to calculate her system. So subsystem A is a mixed state, not a pure one. Not only does the density matrix represent everything Alice can know about her susbsystem, the expectation values of her variables are not at all affected by Bob's subsystem. Even if his subsystem changes, but conserving distinctions, therefore by a unitary transformation U such that

$$\mathbf{U}^\dagger\mathbf{U} = \mathbf{I}$$

Alice's density matrix is unchanged.

$$\rho^A_{aa\prime} = \sum_{bb\prime\prime b\prime} \psi^*(a\prime b\prime\prime)(U^\dagger_{b\prime\prime b}U_{bb\prime})\psi(ab\prime) = \sum_{b\prime} \psi^*(a\prime b\prime)\psi(ab\prime) = \rho^A_{aa\prime}$$

This is what Susskind calls ***locality***: Bob's system, even if it is entangled with Alice's, has no influence on Alice's statistical predictions. This is all about expectation values, in the spirit of quantum mechanics. Adding the least little bit of non-unitary evolution to Bob's subsystem would mean that he could influence Alice's subsystem *faster than the speed of light*.

6. Continuous functions

If a basis of orthonormal state vectors is represented by $|a, b, c, ...\rangle$ for eigenvalues (a,b,c,...), then any state vector may be expanded as

$$|\Psi\rangle = \sum_{a,b,c,...} \psi(a, b, c, ...)|a, b, c, ...\rangle \qquad (23)$$

where the *set of coefficients* $\psi(a, b, c, ...)$ is called the ***wave functio***n of the system in the basis defined by the observables, A, B, C, ….[2] We can expand $|\Psi\rangle$in terms of another set of basis vectors corresponding to different observables and the wave function set will be different even though they describe the same system.

When observables are continuous, the wave function is itself a function of continuous variables. So

$$\sum_i \rightarrow \int_i dx$$

and

$$\int_{-\infty}^{\infty} \psi^*(x)\psi(x)dx = 1$$

Define two linear operators

$$\mathbf{X}\psi(x) = x\psi(x) \qquad (24)$$

and

$$-i\hbar\mathbf{D}\psi(x) = -i\hbar\frac{d\psi(x)}{dx} = \mathbf{P}\psi(x) \qquad (25)$$

which is the momentum operator, the factor $-i\hbar$ being necessary to ensure that **P** is Hermitian and that the units be correct. The momentum eigen-equation then is

$$-i\hbar\frac{d\psi(x)}{dx} = p\psi(x) \qquad (26)$$

2 Some authors use a different definition of the wave function, but this one keeps it separate from the state vector.

of which solutions are of the form

$$\psi(x) = Ae^{\frac{ipx}{\hbar}} \tag{27}$$

The probability for a given value of x is

$$P(x) = \psi^*(x)\psi(x) \tag{28}$$

and for a given value of momentum

$$P(p) = \widetilde{\psi}^*(p)\widetilde{\psi}(p) \tag{29}$$

where $\widetilde{\psi}(p)$ is the wave function in the ***momentum representation***.

The two representations, position and momentum, are reciprocal Fourier transforms of each other:

$$\begin{aligned}\widetilde{\psi}(p) &= \frac{1}{\sqrt{2\pi}}\int dx e^{\frac{-ipx}{\hbar}}\psi(x) \\ \psi(x) &= \frac{1}{\sqrt{2\pi}}\int dp e^{\frac{ipx}{\hbar}}\widetilde{\psi}(p)\end{aligned} \tag{30}$$

As in classical mechanics

$$\{x, p\} = 1$$

so in quantum mechanics

$$\{\mathbf{X}, \mathbf{P}\} = i\hbar$$

which leads to the ***Heisenberg Uncertainty Principle***:

$$\Delta X \Delta P \geq \frac{\hbar}{2} \tag{31}$$

“No experiment can ever beat this limitation.” (270)