# Special relativity

*The minimal minimum*

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## 1. The principles of Special Relativity

Special relativity (SR) is based on two principles:

1. The principle of relativity states that the laws of nature should be the same for all observers in inertial frames (defined below).
2. All such observers, upon measuring the speed of light in a vacuum, will find the same result, c = 299,792,458 km/sec.

The first requirement is necessary in order for physics to be coherent. It means that observers in inertial systems use the same equations. Rather than going on incessantly repeating “in an inertial reference frame”, let’s get it done with once and for all by stating:

> SR considers only observers in ***inertial*** reference frames, those which move with constant (unaccelerated) velocity relative to one another.

The first requirement may be stated differently:

> No experiment can measure the absolute velocity of an observer; the results of any experiment performed by an observer do not depend on his speed relative to other observers who are not involved in the experiment.[1]

The second requirement is a result of rather astounding experimental results. There is no known reason for it, that’s just how it is. Because of the constant speed of light, SR prevents us from considering space and time as being two separate things and explains how they are related and linked into a more global entity, ***spacetime***. More on that in a moment.

## 2. The spacetime diagram and the Lorentz transform

We use the notion of a ***four-vector*** in tensor notation:[2]

> **X** = (t, x, y, z) := ( $x^0$, $x^1$, $x^2$, $x^3$)

or

$$X^\mu = (x^0, x^1, x^2, x^3).$$

We also use standard relativist units, where c = 1, so that

> c = 3x$10^8$ m/sec = 1

meaning that time is measured such that

> 1 sec = 3x$10^8$ m.

We speak of spacetime as the total of space and time, which while different are no longer distinct in SR. An ***event*** is something which happens instantaneously at a specific value of time and space, i.e., t and $\vec{x}$. An event may be simply a set of values of the four spacetime variables. An inertial observer is just a spacetime coordinate system which records the coordinates of any

1 Schutz, 1.

2 More on that in my GR overview.

event. His space is Euclidean (in SR) , distances between given points are independent of time and his clocks are synchronized and run together.

The essential geometry of SR is shown in Figure 1. For simpllicity, we ignore y and z. The black axes show the x and t variables of observer $O$, the blue, of observer $O'$. We suppose that $O'$ is moving with velocity v along the x-direction. The path of the origin of $O'$ as seen by $O$ is the blue line labelled t'. This is because the origin of $O'$ is always at $\vec{x}' = 0$ and so his path is his time axis. From the diagram, the tangent of the upper angle, $\phi$, is given by

$$tan\phi = \frac{dt}{dx} = \frac{1}{v}.$$

Any light wave on such a diagram must move with constant velocity, c = 1, and so is always represented by a 45° line. This fact can be used to show that the x'-axis is the blue line making an equal angle $\phi$ with the x-axis of observer $O$.[3]

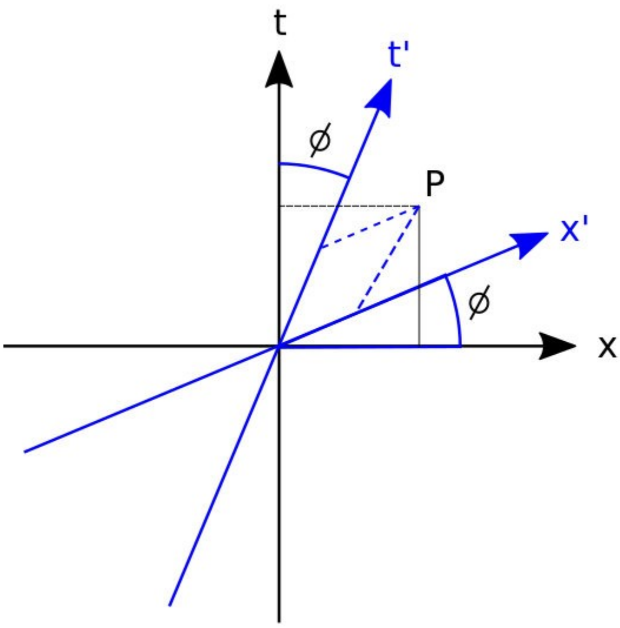


*Figure 1: Geometry of SR – the spacetime diagram (after Collier)*

A point P in space is therefore represented in $O$'s reference frame by the intersections of the fine-dotted lines with the x and t axes. They are obviously normal to t and x. The coordinates of $O'$ are a bit more complicated: Since lines parallel to x' are lines of constant t', and vice versa, the upper dashed line intersects the t' axis at the value t' of P in the $O'$ frame. Similarly, the lower dashed line intersects the x' axis at the value of x' for the point P.

The fact that the speed of light is constant for all observers leads (by means of several sorts of derivations) to the ***Lorentz transformation***s between the reference frames of the two observers:

$$t' = \frac{t - vx}{\sqrt{1 - v^2}} = \gamma(t - vx)$$

$$x' = \frac{x - vt}{\sqrt{1 - v^2}} = \gamma(x - vt)$$

$$y' = y$$

$$z' = z$$

This is often expressed as a transformation matrix:

3 Schutz, 6-8.

$$(\Lambda^{\bar{\beta}}{}_{\alpha}) = \begin{pmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (1),$$

where

$$\gamma = (1 - v^2)^{-1/2} \qquad (2)$$

In fact, this is a Lorentz ***boost***, where one frame moves with a constant velocity relative to another. It may also differ by spatial rotation. This point of view leads to the idea of the Lorentz transformation as a rotation including time, as explained in Section 5.

Then the transformation formula for a four-vector is

$$A^{\bar{\beta}} = \Lambda^{\bar{\beta}}{}_{\alpha} A^{\alpha}. \qquad (3),$$

with $A^{\bar{\beta}}$ and $A^{\alpha}$ expressed as a column vectors. A four-vector is a set of coordinates (displacements) which transform by the Lorentz transformation.[4]

Usiing the Lorentz transform, one can derive somewhat laboriously all the following.

The defined ***interval*** between two events separated by the displacements $(\Delta t, \Delta\vec{x})$

$$\Delta s^2 = -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 = -\Delta t^2 + \Delta\vec{x}^2 \qquad (4)$$

is an invariant quantity across inertial systems. It can be expressed in terms of a metric tensor which is diagonal and has eigenvalues (-1,1,1,1) as

$$\Delta t^2 = \eta_{\mu\nu} \Delta x^{\mu} \Delta x^{\nu}.$$

A space of this kind is called ***Minkowski space***.

The ***proper time*** is also an invariant:

$$\Delta\tau^2 = -\Delta s^2 = \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 = \Delta t^2 - \Delta\vec{x}^2 \qquad \textbf{\textit{(5)}}.$$

There exists in any frame a set of orthonormal ***basis vectors***

$$\begin{pmatrix} \vec{e}_0 \\ \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \textbf{\textit{(6)}}$$

so that a four-vector may be expressed as

$$\vec{A} = A^{\alpha} \vec{e}_{\alpha} \qquad (7)$$

where $\alpha = 0, 1, 2, 3$ and is summed over. Such a four-vector is invariant in spacetime under Lorentz transforms, only its components are different in different frames. The equality

$$\vec{A} = A^{\alpha} \vec{e}_{\alpha} = A^{\bar{\beta}} \vec{e}_{\bar{\beta}}$$

can be used to show that the basis vectors transformation as

$$\vec{e}_{\alpha} = \Lambda^{\bar{\beta}}{}_{\alpha} \vec{e}_{\bar{\beta}} \qquad (8).$$

Comparing this to

$$A^{\alpha} = \Lambda^{\alpha}{}_{\bar{\beta}} A^{\bar{\beta}}$$

shows that the tranformation matrix for basis vectors is the inverse of that for vectors.[5]

4 In GR, a vector is a set of components and *not* a thing which points from one point to another, since that would be impossible in a curved space.

5 In the GR overview, we will see that this means they are not vectors, but one-forms, or dual vectors.

Light rays satisfy $\Delta s^2 = \Delta\tau^2 = 0$ and so move at 45° angles to the t-axis and x-axis (and y-axis) on a spacetime diagram, extended to reprsent two spatial dimensions. Inside the cone along the t-axis, $\Delta\tau^2 > 0$, so an event at the origin has time to influence any point within the cone without going faster than the speed of life. The interior of the cone, for positive or negative t, is therefore referred to as ***timelike***. Outside the cones, this is not possible, so these regions are called ***space-like***.. The cone for t>0 is the ***future*** relative to the origin, the cone for t<0, the ***past***.

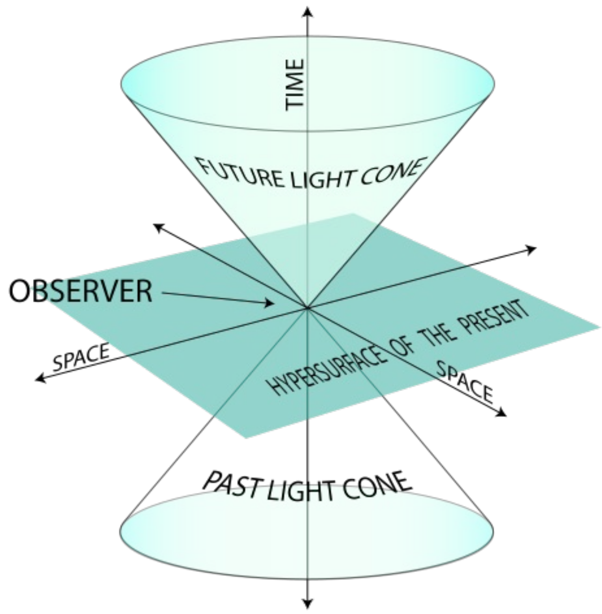


*Figure 2: Light cones (by K. Aainsqatsi via Wikimedia Commons[6])*

# 3. Four-velocity and four-momentum

The ***four-velocity***, $\vec{U}$, of a particle may be defined as a vector tangent to its world line and one time unit long in its rest frame. Observer $O$ sees the origin of $O'$ move along its world line in his frame. In the frame of $O'$, this origin is at $\vec{x} = 0$, so its only non-zero component is along the basis vector $\vec{e}_0$. So the particle's four-velocity is that vector $\vec{e}_0$ in its inertial rest frame, $O'$.

From another point of view, it would be good to define four-velocity as a derivative, as in classical physics. But instead of dt, an invariant quantity, $d\tau$, is used, so the components of the four-velocity $\vec{U}$ are given by

$$U^\alpha = \frac{dx^\alpha}{d\tau} \tag{9}$$

which reduces to $\vec{e}_0$ in the particle's frame. Look at the time component of $\vec{U}$.

$$U^0 = \frac{dt}{d\tau} = \frac{1}{d\tau/dt}.$$

But

$$\frac{d\tau}{dt} = \frac{\sqrt{dt^2 - d\vec{x}^2}}{dt} = \sqrt{1-v^2} := \frac{1}{\gamma}.$$

Then the spatial components are

$$U^i = \frac{dx^i}{d\tau} = \frac{dt}{d\tau}\frac{dx^i}{dt} = \gamma V^i,$$

6 https://commons.wikimedia.org/wiki/File:World_line.svg.

where $V^I$ is the ordinary velocity. So

$$\vec{U} = (\gamma, \gamma\vec{V}). \qquad (10).$$

Note that

$$U^2 = \vec{U} \cdot \vec{U} = \gamma^2(-1 + v^2) = -1 \qquad (11),$$

by the definition of $\gamma$. This is also an invariant.

The ***four-momentum***, $\vec{p}$, is defined, similarly to classical mechanics, by

$$\vec{p} = m\vec{U}, \qquad (12)$$

where m is the particle's mass in its own frame. So

$$\vec{p} = m\gamma(1, \vec{V}) \qquad (13)$$

and

$$\vec{p} \cdot \vec{p} = m^2\vec{U} \cdot \vec{U} = -m^2$$

so $p^0 = m\gamma$ is the particle's energy and $\gamma\vec{V}$ its three-momentum. Also,

$$\vec{p} \cdot \vec{p} = -E^2 + p^2 = -m^2,$$

where $p^2$ refers to the three-momentum, and so

$$E^2 = m^2 + p^2. \qquad (14)$$

Note that when v << c, the energy

$$E = p^0 = m(1 - v^2)^{-1/2} \approx m + \frac{1}{2}mv^2,$$

the rest-mass energy plus the kinetic energy of classical mechanics.

If particle p' moves in the frame of p with velocity v along the x-axis, and a particle is moving with velocity W along the x'-axis of p' , then its velocity in p's frame is given by the ***Einstein law of composition of velocities***:

$$W' = \frac{W + v}{1 + Wv} \qquad (15)$$

which can never exceed the speed of light (1, in this system).

# 4. Curious results of SR

Consider this spacetime diagram, in Figure 3 with an invariant hyperbola, $\Delta s^2$, about the t-axis.

All points on the hyperbola[7] have the same (invariant) interval or proper time, so point A is at 1on the t-axis and point B is at 1 on the t'-axis. The red dotted line is tangent to the curve at B and parallel to the x' axis and is therfore a line of simultanteity for O' at constant t'=1. The horizontal dotted line through A and C is a similar line of simultaneity for O at constant t=1. What can this tell us?

O' sees point D lying on its line t'=1. But O sees D at t<1. So O' thinks the clocks of O are running slow.

7 It's really a parabola, since I could not figure out how to make a hyperbola in Inkscape. Hope nobody notices.

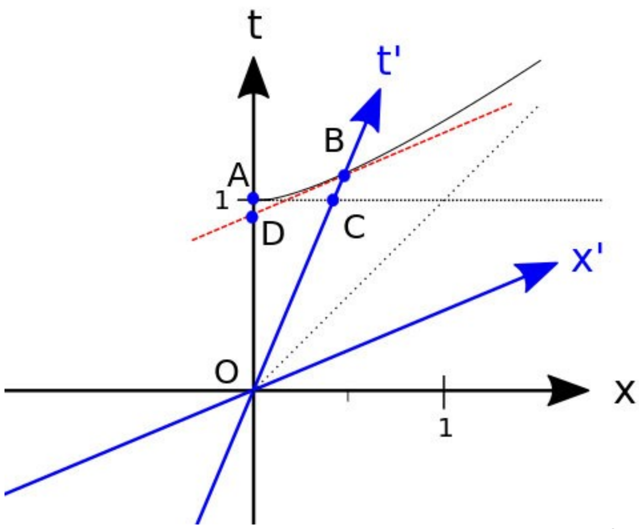


*Figure 3: Time dilation (after Collier[8])*

Inversely, O' sees event C as occuring at time t'<1. But O sees it at t=1. So O thinks the clocks of O' are running slow.

Using the Lorentz transform, one can calculate the quantity of this ***time dilation*** as

$$(\Delta t)_O = \gamma(\Delta t'_O) < \Delta t'_O.$$

If the diagram of Figure 3 is rotated 90° clockwise, similar considerations show that each observer thinks the other's distance is contracted in a direction along its trajectory. This is ***length contraction***.

Several so-called paradoxes, which are in fact incorrect ways of framing the situation, are well known.

Time dilation leads to the so-called ***twin parado***x: If one of two twins moves away at great speed, then reverses direction and returns, she will have aged much less than her earth-bound twin. The reason why the inverse is not also true (aside from the impossibility of it) is because she undergoes acceleration and so is not always in an inertial system. Further consideration falls into the realm of GR.

Length contraction leads to the problem of the pole in the barn and shows that simultaneity must be abandoned. Leonard Susskind has updated this problem to a stretch limousine and a garage for a VW bug. As seen by an observer stationary relative to the garage, which has doors at both ends, the limo, if it moves at a speed close to that of light, will be contracted so that it might fit all into the garage at once. In particular, O will see the following sequence of events:

1. Limo front enters garage front door;
2. limo tail enters garage front door;
3. limo front leaves garage back door.

But the limo driver rather sees the barn as being contracted, so there is no way he can fit into it all at once. He sees the following sequence:

1. Limo front enters garage front door;
2. limo front leaves garage back door;
3. limo tail enters garage front door.

Note the reversal of the sequence of events 2 and 3. Simultaneity is out the door!

8 All the reasoning concerning this diagram comes from Collier, 123-4.

## 5. The Lorentz transformation as a hyperbolic rotation

The Lorentz transformation may be seen as a rotation involving the time coordinate. For a spatial rotation, a typical Lorentz transformation matrix might be the following:

$$(\Lambda^{\bar{\beta}}{}_{\alpha}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & cos\theta & sin\theta & 0 \\ 0 & -sin\theta & cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

So a rotation including time may be written

$$(\Lambda^{\bar{\beta}}{}_{\alpha}) = \begin{pmatrix} cosh\phi & -sinh\phi & 0 & 0 \\ -sinh\phi & cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The $\bar{O}$ frame ($\bar{x} = 0$) is then moving with velocity v such that

$$\bar{x} = tsinh\phi + xcosh\phi = 0$$

and

$$v = \frac{sinh\phi}{cosh\phi} = tanh\phi.$$

Then

$$1 - v^2 = 1 = \frac{sinh^2\phi}{cosh^2\phi} = \frac{1}{cosh^2\phi},$$

since

$$cosh^2\phi - sinh^2\phi = 1.$$

So

$$\gamma = \frac{1}{\sqrt{1 - v^2}} = cosh\phi$$

and we get back to the usual form of the Lorentz transform.

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