Quantum Field Theory in words

What modern physics says about the nature of reality

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Being one more attempt to convey how modern physics explains the Universe. The language of physics is mathematics, so rather than invent analogies, I will try to explain it from the math, but using words, not mathematical symbols or equations.

1. The language of physics

To repeat: Mathematics is the language of physics. That's the main point, so you can skip to the last paragraph of this section if you so desire.

First assumptions. Without going into ontology, we will assume simply that there is a reality "out there" to observe. Understanding it will only be possible if it is governed by universal natural laws and so we shall assume their existence also.[1] In order to deduce these laws, we observe the phenomena of nature.

We speak of a ***law*** when we have observed that a given event (called the cause) in certain specific and well-defined conditions always gives rise to the same event (the result) in countless observations. The word "always" here means "every time we have measured it". So we consider that this will always be true in the future too. This is a second philosophical assumption we always make, after the one that what we observe with our senses (and instruments) really exists in some meaningful and useful way.

Observation, with or without advanced equipment, proceeds when our sense organs are activated by numerous signals which are flowing and streaking around us (light waves, chemical substances, molecules of all sorts, and more.). We must not forget that only a rather limited subset of those signals are able to trigger our senses. They then send electro-chemical impulses through nerves into our nervous system, usually into our brains, where they are collated and linked into meaningful and useful forms. Don't ask what that means in terms of consciousness, because nobody knows – yet.

1 The French, ever alert to a nuance, refer to them as principles.

The word “useful” here originally referred to information which helped our remote ancestors to survive. But later, as we evolved through natural selection, so did our cognitive processes and our understanding of the world around and within us. Then we started using language to express and exchange our ideas. Spoken and, later, written language constitute a conceptual system we use to express and communicate our thoughts. Another one is mathematics.

The notion that the language of physics is mathematics can be approached from a somewhat different angle. Our languages are tool kits which admit expression of some ideas better than others. Eventually, we discovered new phenomena which our former vocabulary was not able to describe, so we invented new words and concepts. Our ancestors needed words like tree or tiger, they did not need gene or quark. Those organisms which developed language to help them cooperate, not only to defend themselves but to come up with new ideas for survival or simply to make life more pleasant, lived to reproduce. So languages evolve, hopefully to express useful or beautiful ideas more precisely or … beautifully.

As our languages and concepts evolve, they may try to make use of older concepts by extending their meaning or putting them together in new ways. This has led, for instance, to our considering quantum objects to behave like waves and particles at once. Maybe It would be better to have a new term for something which is neither – or both! What do *you* think? If that word were, say, wavicle, would that aid in our description of nature, once we assimilated the word the way we have assimilated the words wave and particle?

Furthermore, language is a cognitive function which is therefore limited by the structure and functioning of our brains. To get to the point, just as languages are imperfect but evolving means of expression, maybe our mathematics is also imperfect. It’s certainly evolving, along with our concepts and paradigms. Just compare Newton’s equations to Einstein’s or those of quantum mechanics.

Newton (force): $F = ma$

Newton (gravitational force): $F = G\dfrac{mM}{R^2}$

Einstein (gravity): $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = -\kappa T_{\mu\nu}$

Schrödinger (QM): $i\hbar\dfrac{\partial\Psi(x)}{\partial t} = -\dfrac{\hbar^2}{2m}\nabla^2\Psi(x) + V(x)\Psi(x)$

Dirac (QFT): $(i\gamma^\mu\delta_\mu - m)|\psi\rangle = 0$

(Dirac’s equation may look simpler than Schrödinger’s, but trust me, it’s not.) This evolution suggests that one day our equations may get still better and approach even closer to describing reality precisely, i.e., become better approximations.

The math of physics has become quite hairy, as is understanding what it means in terms of physical phenomena. Some of us may have just reached the point where we can almost imagine everything around us as made up of particles separated by relatively vast stretches of empty space. But now, physics tells us that everything is be made up of fields. This is the idea behind ***quantum field theory*** (***QFT***). Fields are where the buck stops, at least for the moment.[2] Like the bottom turtle, they are not made up of anything else (well, as far as we know).[3] The particles which we see as the constituents of all the stuff around us are vibrations in quantum fields, fermion fields for matter, boson fields for forces. If you have trouble imaging a proton field interacting with an electron field through a vector boson field, you are not alone. It’s much easier to imagine them as particles, so that’s what we do. Is that because our brains are built to comprehend particles better than fields? Or just habit? Or the problem of representing our ideas with language?[4]

Our understanding is constantly evolving. We labored for a long time with Newton’s relatively simple equations, then added Maxwell’s more complicated ones, then Schrödinger’s and Einstein’s. Since then, we have added math evolved by Dirac, Weinberg, Salam, Higgs, Feynman and many others. The language is math, but – just like English or French – it’s evolving in terms of what it expresses.

2 We are ignoring far-out ideas like strings, branes and loops.
3 Some scientists think that space itself is made up of elements. This is the theory of loop quantum gravity.
4 Is language necessary for thought?

We must avoid going Platonic here. There is neither cave nor any reason to consider that we are looking at shadows of the Ideal, whatever that might mean. In spite of our initial assumptions, we don't know what's out there. Which doesn't keep us from trying to cope with it. Especially since it seems to work so well. The results are quite astounding in their accuracy and precision. But we must not forget that our perception of the Universe is limited both by our sense organs and by the structure of our brains.

2. Transformations and constraints of Special Relativity

Physicists can't take on the whole Universe at once, so they study specific, isolated systems. Don't be put off by the word ***system***. It just indicates the phenomenon or phenomena we are studying, usually considered as isolated from the rest of the Universe. This method of considering a limited, isolated collection of objects, called ***reductionism***, is decried in some quarters, but it works pretty damn well.

What concerns us now is, what are the things you can do to such a system without actually changing the system itself. You can rotate it in any direction, or move it from here to over to there or look at it in a mirror. If the system remains unchanged under these ***transformations*** (physics speak), then it is said to be ***symmetric***, or ***invariant***, under the transformation.

Some examples. You can rotate a sphere, say a baseball through any angle in any direction and, if you ignore the seams, it still looks the same: This is spherical symmetry. You can rotate a candle through any angle around its lengthwise axis and it looks the same: This is obviously cylindrical symmetry. A cube is more special; you can rotate it through an angle of of 90° around an axis through the mid-points of two opposite sides and the result is indistinguishable from the original. A pancake is either up or down, not in between. Although we are talking about the system's looking the same, the important point is that this requires that the equations which define it be the same before and after. And that helps us with the math, by eliminating non-symmetric forms of the equations

The transformations just mentioned are static, each happens once then is finished. But you can make dynamic (moving) transformations too. You can look at the system from a moving vehicle like a train. As long as the observation vehicle moves with a constant speed and direction relative to the system it's observing, or vice versa, the equations describing the system must look the same. Such a constant-speed, unaccelerated observation platform, or ***reference frame***, is called an ***inertial system***. The changes due to going from, say, a stationary reference frame to an inertial, moving one are called ***Lorentz transformations***, part of the theory of ***Special Relativity*** (***SR***) published by Einstein in 1905 (just to give an idea of the time frame involved). In addition to such ***boosts***, due to relative motion at constant velocity, Lorentz transformations also include ordinary rotations.

SR has a real surprise for us and it is crucial: It says that whether you are sitting still (relative to something) or moving inertially, then if you measure the speed of light, you will always get the same result. To the nearest meter per second, this is 299,792,458 meters per second (mps), but we will take the approximate value of 300,000 km/sec. Think: If I am moving past you in my car at 100 Km/sec and a really fast TGV Is moving past you at 200 Km/sec in the same direction, I will measure the TGV's speed relative to me as 100 Km/sec. This is intuitive. But If I am moving at half the speed of light past you, 150,000 Km/sec, and a photon (light particle) is moving past me, both you and I will measure the photon's speed as 300,000 Km/sec. I will not think it is moving at 150,000 Km/sec. This is not intuitive but it is true, and it has been confirmed in countless experiments.

> According to SR, *the speed of light (In a vacuum) is constant* – always the same, as long as you measure it from an inertial (non-accelerating) reference frame.

Based on this "law", we actually can derive quite easily the equations of the Lorentz transformation by using only algebra and simple geometry. Physicists then say that in order to be valid, their equations all must be invariant under Lorentz transformations.

The invariance of the speed of light has important consequences. We measure speed in, say, kilometers per second, or km/sec, length divided by time. This means that the invariance of the velocity of light imposes a constraint on the relation between length and time. The result is that we can no longer consider physical events taking place in space and time separately, but only in a four-dimensional ***spacetime***. Fortunately this constraint is not very important for velocities well below that of light, which is why we managed quite well without it for a long time, as in the example with the TGV.

In addition to the two assumptions about reality and laws, we now have acquired three no-longer-new requirements.

1. The speed of light in a vacuum is constant, always measured by an observer in an inertial frame of reference to be the same.
2. The equations of physics must be invariant under Lorentz transformations – rotations and boosts. This is referred to as being ***covariant***. It means the equations which describe nature must take the same form before and after the transformations. This Is so important I will restate it: The laws of physics must be Lorentz-invariant.
3. The invariance of the speed of light imposes constraints on the relation of space and time which mean we must understand them not as two concepts, but connected together as spacetime.

Mathematically, the equations of the Lorentz transformation show clearly that we can no longer take space and time to be independent. A Lorentz boost changes not only where the studied object is (its spatial coordinates) but also how long it is there (its temporal coordinate). It therefore also changes velocities. And it means that space and time can no longer be considered as independent, so we must consider the four coordinates of time and location together as what's called a ***4-vector***, usually written in the order (t, x, y, z).[5] The equations lead to fun facts like these:

- The faster you go relative to somebody else, the slower he thinks your clocks run, including your internal body clock. This leads to the twin paradox: A twin who takes a joy ride in a space ship comes back younger than one who stays at home on Earth. This was portrayed quite vividly in the movie "Interstellar". Note that the errant twin must accelerate up to speed and then stop (i.e., decelerate) and turn around to come back, so she is not in an inertial state of constant relative motion. That is why she does not think the twin who stayed home is younger than she is.
- The faster you go, the thinner you get along the direction of motion. A result of this is that ...
- … it is often impossible to say whether one event occurred before or after another, which means that the notion of simultaneity is no longer tenable.
- I can't keep myself from jumping ahead and pointing out that Einstein's theory of gravity, ***General Relativity*** (***GR***), says that clocks run more slowly in a stronger gravitational field. This means that a clock on Earth at sea level runs slower than one on the International Space Station – or on a GPS satellite. You might want to say that gravity slows down time, but that's not really true. All clocks run at one second per second in their own frame of rest (where they are not moving, meaning x, y and z are constant and only time changes).

Here's a fun example about that subject of simultaneity. Leonard Susskind has updated the classic example of a pole vaulter and a barn to a stretch limousine and a garage for a VW beetle. As seen by an observer who is stationary relative to the garage, which has doors at both ends, the limo, if it moves at a speed close to that of light, will be contracted so that it might fit all into the garage at once. In particular, the observer will see the following sequence of events:

1. Limo front enters garage front door;
2. limo tail enters garage front door, because the contracted limo can fit in the garage;
3. limo front leaves garage back door.

But the limo driver rather sees the barn as being contracted, so there is no way he can fit into it all at once. He sees the following sequence:

1. Limo front enters garage front door;
2. limo front leaves garage back door;
3. limo tail enters garage front door.

Note the reversal of the order of events 2 and 3. Time ordering and, so, simultaneity are out the door (of the garage)!

5 4-vectors are manipulated mathematically using tensors and matrices, so the necessary amount of math just increased.

3. Fields: electric, magnetic and gauge

Physics is not just about moving objects (***mechanics***), it's also about electricity and magnetism, together constituting ***electromagnetism*** (***EM***). This subject was elucidated already in the 19th century by James Clark Maxwell, whose famous equations describe all of EM. Maxwell published these equations in 1861, 44 years before Einstein published SR. Maxwell's equations possess two amazing properties.

1. Maxwell's equations can be solved as a wave traveling at a speed which is explicitly given in terms of two well-measured quantities and calculation shows it to be 300,000 Km/sec – the speed of light! And there's just one such speed. So Maxwell prefigured Einstein.
2. Maxwell's equations are already Lorentz invariant. (Newton's are not.)

Maxwell's equations also confirmed one more concept which was to become primordial for modern physics – the concept of a field.

We know that EM talks about electricity and magnetism and so about electric and magnetic fields. These fields had been discovered by Michael Faraday in 1821. A ***field*** is a physical quantity which has a value at every point in space and time. It may be a ***scalar*** which simply has a value everywhere, like temperature, about 22°C indoors as I write and about 6°C outside. Or it could be a ***vector***, meaning it has a direction as well as a value. An example of this would be the wind, which here now has, say, a value of 25 Km/sec and a direction of westerly, meaning it blows from west to east. Electric and magnetic fields are also vectors. The electric-field vector points from one electric charge towards or away from another, depending on whether they are of opposite or the same signs. The magnetic-field vector points from the south to the north pole of a horseshoe magnetic. (The particular directions are conventions, they could be reversed and all would be well.)

Maxwell found both the two EM fields could be defined in terms of two other fields, one a scalar and the other a vector. Together, they constitute another 4-vector, a thingy in 4-dimensional spacetime which transforms by the Lorentz transformations. So we can think of EM as really about one thing, a 4-vector called the ***EM vector potential***.

One can do a transformation of the EM potential which is said to be ***local***, meaning the change is not constant everywhere. Until now, we have considered transformations which were the same in all of space. A rotation, for instance, can be described by a single angle of rotation, so it is considered to be a ***global transformation***.[6] Imagine (if you can) rotating one corner of a cube through 90° about some point and another corner through only 45° about some point. Our poor cube would not be much of a cube any more.

Here's the biggy. It turns out that there are types of local transformations (not the same everywhere) of the EM vector potential which do *not* change the results of Maxwell's equations! This is because the vector potential itself is not something we can measure. We can only use it to calculate the electric and magnetic fields, and then we can measure them. In the jargon, we say that Maxwell's equations are ***invariant*** under ***local transformations*** of the EM vector potential.

For historical reasons, the EM vector potential is called a ***gauge field*** and its allowed transformations are ***gauge transformations***. Think of trains in the old days, when different countries had different distances, called gauges, between the rails. The passengers didn't notice the gauge – once they had changed to appropriate cars. This is likely the origin of the term gauge in physics. Like it or not, the word "gauge" remains to describe one of the most important concepts in modern physics.

To take home: We can have identical physical situations for different values of the gauge fields. This is because we can not measure gauge fields, only the other, physical fields, which are calculated from them. The fact that different gauge fields give the same measurable field is a redundancy in our description of nature – two things which give the same result.

Just to whet your appetite, here is where we are going. *All* the four forces of physics are due to invariance under local transformations of gauge fields. Stay tuned...

Notice that so far, we have not used the dreaded word "quantum". It's time now to take the plunge.

6 Remember, rotation means that all objects are rotated about a common point, the center of rotation.

4. Operators in Quantum Mechanics (QM)

Now things get hairier, so you may want to rest now with a hot cup of coffee … or a cold beer. Up til now, we have discussed what is called ***classical*** physics. In physics, the word “classical” has nothing to do with style or expressiveness; there is no such thing as “romantic” physics.[7] But as soon as we bring in QM, we are not talking about classical physics any more. I don’t know how to explain what follows without at least talking about equations. Actually, I’ve already started that, haven’t I?

You probably have heard that on a microscopic scale, things we measure no longer have a continuous range of values. For instance, instead of taking on any value between 1 and 10, say, 7.39, they may only have integral values: 1, 2,... 10. In fact, measured values of parameters aren’t necessarily continuous on a macroscopic scale either, but the effects are so tiny we don’t notice them. However, on a microscopic scale, such quantum phenomena are proportionally greater and clearly show evidence of the granular nature of things. For instance, the energy of an electron in an atom can only take on certain distinct, separated values, the differences of which are seen in the spectrum of colors of light they emit. When you heat a piece of iron, like the coil on an electric stove, it glows red because the electrons In the atoms are radiating (almost) only with an energy corresponding to the color red. These phenomena are things that physicists can observe and measure. But we want to have an explanation of why or, at least, how this comes about.

Microscopic and sub-microscopic phenomena like this are explained by using ***quantum mechanics***. ***QM*** is a set of rules for calculating things and is quite general in scope. In order to study a particular phenomenon, we must furnish more information: the properties of the system in question, the forces at play and any initial conditions. These might be the initial positions and momenta and the masses of two elementary particles. There are two ways of getting to the quantum version of phenomena:

- If classical equations for the domain exist, we can quantize them.
- Otherwise, we are forced to invent the quantum version. In order to do this, we use clues and tricks, one important one of which is the symmetry we have talked some about. We will use this method in QFT.

Important, even crucial point: The very basis of QM is based on the notion of *commutation* rules for *operators*. What do those two things mean?

In classical physics, we measure things like mass, position and velocity. In fact, physicists prefer ***momentum***, which is the product of mass and velocity in low-energy, non-quantum, classical physics. How do we go from classical equations of continuous energies to quantum ones? Mathematically, what we do is modify our equations by replacing those physical measurables, position and momentum, by ***operators***. These are not like telephone operators, if you remember them, but more like surgeons, who operate on someone, thereby changing them. The process of defining operators in place of simple variables is called ***quantization***.

The position and momentum operators have the curious but essential property that if you apply them in one order, say position before momentum, you won’t get the same result if you measure them in the opposite order, momentum first. Physicists say these operators do not ***commute***. This is one of the fundamental characteristics of QM. Similar results hold for other pairs of quantities, like energy and time, which also depend on their order of measurement. Many physicists consider these ***commutation relations*** to be not a result but the very *basis* of QM. Then the difference between the two orders of operation is a very tiny number whose magnitude is equal to Planck’s constant.

This is as good a moment as any to make a brief aside to acquaint you with two important numbers used in QM. The first is called, both by physicists and mathematicians, i and is equal to the square root of -1. Yes, minus one. You may protest that such a number cannot exist, it is not real. We agree, it is not real, so it is called ***imaginary***. In math, it allows the construction of imaginary quantities and we know how to manipulate them as so-called complex quantities. The word ***complex*** here does not indicate complicated, though in some cases it is that too. It just means containing imaginary quantities, i.e., including

$$i = \sqrt{-1}.$$

The fact that it is not real allows us to employ math ideas without which we just could not do QM. The other number is ***Planck's constant***, h, which is usually divided by 2π and written $\hbar$.[8] The difference of the non-

7 There is no such thing as “woke” physics either, although individual physicists may be more or less woke.

8 Planck’s constant is equal to $6.62607015 \times 10^{-34}$ J·Hz.

commutating variables is just the product of i and $\hbar$. Both the momentum operator and the Schrödinger equation cited above contain both these quantities. In spite of this, all our measured quantities, including momentum and energy, turn out to be real, not imaginary.

Some everyday examples, really analogies, of non-commuting operations might be filling and emptying a cup of coffee. Filling and then emptying does not give the same result as emptying and then filling; only the latter ordering leaves you a delicious brew to drink. Simple math gives examples like addition and multiplication. Suppose I act on the three numbers 2, 4 and 6 by adding the first two and multiplying by the third: That gives (2+4)*6 = 6*6 = 36. But if I multiply first, I get (2*4) + 6 = 8 + 6 = 14, not at all the same result.[9] Ordering is important.

All very well and good, but if position and momentum are operators, what do they operate on? The answer was published by Erwin Schrödinger in 1926, thereby assuring him a Nobel Prize as well as having his picture on post-war Austrian banknotes until he was bumped aside by the generic (and fake) architectural designs of euro notes. But I digress... Schrödinger derived an equation in which the operators operate on a thing called the ***wave function***. In the same year, Max Born showed that the absolute square of the wave function could be interpreted as the probability that the system studied was in a given state. And so QM became probabilistic. The wave function was therefore referred to as a ***probability amplitude***, called an amplitude because it must be squared in order to give a probability.

If such operators operate on special wave functions called ***eigenfunctions***, the result is simply to return the wave function multiplied by the value (the ***eigenvalue***) of the quantity represented by the operator. These eigenvalues are the set of discrete, generally discontinuous values the quantity represented by the operator is allowed to take on. It's really simple:

(operator for X) x (eigenfunction of X) = (eigenvalue of X) x (eigenfunction of X).

The message to take home is that assuming physical quantities like position and momentum to be operators ***quantizes*** them and renders them non-commuting. Then the Schrödinger equation only has solutions for discrete (discontinuous) values of parameters like energy.

About now, you may be wondering where this wave function is. Answer: In an abstract mathematical space called a Hilbert space.[10] Much more math is required in order to explain that, so we'll skip over it and go on sticking to words. Just be aware that physics proposes the existence of many spaces, for spin, but also for isospin and other quantities. These are internal spaces though, we cannot go into them from our 4-d spacetime. We'll get to that subject In a moment.

5. Symmetry in QM: spin

What we have discussed so far is standard, non-relativistic QM, meaning that it is not covariant under Lorentz transformations. Now we have to apply what we know about symmetry under transformations . The first step is to require the equations we use to be compatible with SR, i.e., Lorentz-invariant. The equations should be the same if we look at the system after rotating it or when observing it while we are moving inertially, with constant velocity. Now we are working in relativistic QM, or RQM. We then come up with not one, but three possible equations for describing free particles. (Don't worry, you don't have to remember these.)

- one for particles with spin zero, the Klein-Gordon equation;
- one for particles with spin ½, the Dirac equation;
- and one for particles with spin 1, the Proca equation.[11]

Oops, did I say spin? Uh... an explanation is obviously in order.

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9 A still different result comes from 2*(4 + 6) – 2*10 = 20.

10 Much of QM concerns abstract objects which only exist as mathematical concepts or formulas. But images (like analogies) of these abstract objects can be set up, or imitated, in real spacetime in such a way that the real objects behave like the abstract-math ones. Such analogical objects are called ***representations***. If that's too much, don't worry about it.

11 When we say a spin value of, say, 1, we mean one unit of spin measured in quantum terms as a multiple of $\hbar$, the Planck constant divided by 2.

But first, why bother? We will see reasons below, but an important one comes from a theorem published by *mathématicienne* Emmy Noether in 1918. The theorem explains the link between symmetries, like those we are considering, and conservation laws. Conservation of X means that we can measure X at any point in the life of the system and it will always be the same. ***Noether's theorem*** says that symmetries give rise to conservation laws. Specific examples include the conservation of momentum, due to symmetries under translation (displacement); of angular momentum, due to rotational symmetry; and of energy, due to symmetry in time.

Spin is a sort of angular momentum. In the absence of forces, like friction, momentum is conserved, meaning you don't slow down, but coast along forever. Such momentum is more commonly referred to by non-physicists as inertia. A similar thing is true when you are not moving along in a straight line, but rotating, like a spinning ice skater. Yes, I said rotating, like we were talking about at the beginning of section 1. Angular momentum is conserved too, meaning that in the absence of forces, you just keep on turning. It's the conserved (constant) angular momentum of your rotating bicycle wheels which helps them keep their (and your) upright position, rather than falling over. You can see similar behavior in a top, before it slows down and falls over. A gyroscope also functions by conservation of angular momentum. Now we can't say that spin is angular momentum, because we can't see anything spinning, nor does the theory suggest anything. But the equations for spin behave just like those for angular momentum. In fact, they are identical in their form. It's just that they are not for ordinary angular momentum, but for something else, which is called ***spin***.

Now hold on tight. As I said, we cannot see what is turning to produce spin. In order to explain this angular-momentum-which-is-not-angular-momentum, we suppose that it is not turning in the same 3-dimensional space as angular momentum like that of our bicycle wheel. It is in its own separate space, a space called an ***internal space**,* as opposed to the exterior space of spacetime. I'll pause a moment to let that sink in.

…

Yep, there are other spaces than the 4-d one we think we live in. Spin occurs in such another space, which can have different dimensions than the 3-d one we all know and love or the 4-d spacetime of Einstein. Although spin lives in another space, it can have effects which are observable in our day-to-day spacetime. Since it is in another space from ours, we cannot measure it directly but simply infer it from experiments, for instance by its effect on a particle's behavior in a magnetic field. If the spin space has no dimensions (imagine that!), the particle described is a ***scalar*** with spin 0. If the spin space is 2-d, the particle is a ***spinor*** of spin ½. If it is 4-d (really twice 2-d), the particle is a ***gauge boson***.

If you prefer, you can think of these other, internal spaces as just convenient interpretations of the math, which looks like that for a space but in terms of other quantities than length, time, momentum and so on. But I find the Universe of all those spaces to be far richer, indeed, quite fascinating. If the equations look like the equations of space and are transformed (more or less) like equations of space, chances are they represent a space. Your choice.

And yet, the spinor space is not completely independent of "ordinary" spacetime. Indeed, one can show (but I won't) that a rotation in spacetime through a certain angle corresponds to a rotation of half that angle in spinor space.[12] In particular, what we see as a difference of up an down in ordinary space (180°) corresponds to a change through only 90° in spinor space. Hmm...

There is yet another way of looking at this. Particles whose spins are an odd multiple of ½ (½, 3/2,...) are ***fermions***. Particles whose spin is an integral number – 0, 1, 2, etc., – are ***bosons***. The main thing here is the distinction between fermions of non-integral spin and bosons of integral spin. The existence of these different values of spin come out of the theory of ***Lie groups***, which describe transformations like rotations or Lorentz transformations, the latter being the source of spin.[13] Classical angular momentum is due to symmetry when rotations are done in normal spacetime; spin comes from rotations in spin space.

You may now skip the next paragraph, but if you do, you'll miss something.

If you are still with me, the group which describes Lorentz transformations is in fact a product of two groups, each having either zero or half-integral spin. Since pairs made up from combinations of 0 and ½ can give 0, ½ or 1, these are the allowed values of spin for the particles represented by the three equations already mentioned.

12 This is too complicated to derive here. See my document on symmetry and QFT.

13 The Lie algebra, an instantiation or example of the Lie group, for the Lorentz transformation is really a pair of rotational (or unitary) groups (SU(2), since you asked), each in 2-d, for a total of 4-d. You don't need to know that.

No experiment has discovered an elementary fermion with spin other than ½, although many composite particles exist with spin 3/2, 5/2 and so on.

While we're on the subject of fermions (remember, half-integral spin) and bosons (integral spin), you will be interested to know that:

- Fermions are the particles which constitute matter. This is because, according to the Fermi exclusion principle, no two fermions are allowed to occupy the same state. If they could, they would, and there would exist absolutely no structure of any kind, such as that of rocks, plants or us.[14]
- Bosons are the particles which carry forces. (We will see why shortly.)

Now on to stuff yet farther out.

6. On to QFT

Welcome back. A quick review:

- According to SR, the speed of light in a vacuum will always be measured by an observer in an inertial (non-accelerating) reference frame to be the same, 300,000 Km/sec. This fact prohibits our considering space and time as separate, but requires us to work in 4-d spacetime.
- SR requires the equations of physics to be Lorentz invariant, unchanged under Lorentz transformations, which may be rotations or passage from one inertial system to another. Maxwell's equations of EM are already Lorentz invariant.
- Particles have spin of values 0, ½, 1 or other multiples of ½. Elementary fermions, the stuff of matter, only have spin of ½. This spin behaves like an angular momentum, but in its own internal space.
- Particles with half-integral spin are fermions and compose matter. Particles with integral spin are bosons and carry forces.

Stated thus simply, that's not so hard. The next step can follow one of three different paths, all of which are valid and useful, some more so in certain situations.

6.1. Second quantization – many particles

Using all this and symmetry, we can deduce the equations of motion for three types of *free* particles: scalar (spin 0), spinors (spin ½, fermions) and vector bosons (spin 1). By "free" particles, I mean with no other particles or fields around to complicate things. It's as if they were all alone in the Universe. Having written the equations, we can solve them. The solutions are fields which are expressed as sums of terms, each one multiplied by a "weight" parameter. The "terms" themselves are plane waves of differing energy and momentum. They behave a lot like harmonic oscillators, such as pendulums. By "behave like", I mean their equations are similar. This is nice, because physicists have been studying these beasts for a long time and understand them really well. But studying one lone particle, with no others about to drop in and chat, is not very interesting for the particle or for us. So we need to go further.

I don't know who thought of this, or how, but the trick is to repeat what we already did in order to get to QM, where we quantized the position and momentum by converting them to operators. The trick is to do the same thing again, but this time, we quantize the free-particle *fields*, the solutions to the three equations of motion. Since each field is a sum of terms, we must also quantize the coefficients of the terms, meaning we treat them as operators. In the case of position and momentum operators, the result of using two of them will depend on their order and this is responsible for the next happy result: We discover that the quantized coefficients do not all commute: The result of applying one of them after another depends on their order. Because of this property, they now can be added together or multiplied (no more complicated than that) in ways which make them operators for the counting, creation and annihilation of chunks (quanta) of energy. And hey, we can take these energy quanta as … particles! So this trick, logically called ***second quantization***, allows us to express the results of field theory in terms of particles and gives us a mathematical technique for creating or destroying any number of them. This

14 Again, this is too complicated to go into here. Suffice it to say that the commutation relation for fermions is not the difference in applying two operators in opposite orders, but the sum.

is a great improvement over equations for a single particle.

As Matthew Schwartz says: “At the risk of oversimplifying things things a little, that [second quantization] is all there is to quantum field theory. The rest is just quantum mechanics.”[15]

Our original free particle needs no longer to feel rejected and abandoned. It has company now. Note, though, that all its new friends are just like it, the same kind of particle. Not just social interaction, but some variety would be nice.

6.2. Interactions – local transformations and gauge invariance

It turns out there is another trick we can use, based on symmetries. We already know that the equations of motion of the particles are Lorentz invariant because we wrote them that way. But that was global Lorentz invariance, the same everywhere. Taking the gauge invariance of EM as a clue and remembering that rotations are a subset of Lorentz transformations, we can look to see what happens if we make our equations *locally* invariant under appropriate rotations. The rotations in question are called ***unitary*** transformations and they have the property that they conserve a type of wave-function product called an inner product. Inner products are used to calculate the probability amplitudes for the different states of the system, so this is a Good Thing, since we do not want the observable properties, even if they are only probabilities, to change. Invariance, remember?

An essential component of our mathematical toolkit is something called a ***derivative***. A derivative calculates the rate of change a quantity. That means how much one quantity is altered as another quantity changes. The latter quantity is often time. Acceleration, the rate of change of the speed of your car, for instance, is the difference in speed at one moment and the next divided by the time elapsed between the two moments. That is a time derivative, because you have a difference in time in the denominator of the calculation. If you are walking in the mountains, the slope you are struggling to conquer is the difference in the altitude divided by the horizontal difference between the two points. That’s a space derivative, since the denominator is a distance in space. In QM, we have both types, because we are using 4-vectors which measure space and time and we want the rate of change with respect to both.

The problem is that when we take a derivative of our locally transformed function, we can’t calculate its change just by taking the difference of the values of the function at the two points. We must also take into account the fact that the transformation is different from one point to the other. Rather than transform the difference, we must do something to the function at, say, the second point in order to make it as if it were evaluated at the first point, where the transformation is the same. Only then can we take the difference. This calculation can be done and the result adds an extra term to the derivative. Since this term allows us to connect the state of the function at one point to its state at another, it is called a ***connection***. When it is added to the derivative, the result is called the ***covariant derivative***. The equation in terms of the covariant derivative is covariant too, meaning it does not change under the local transformation. With an ordinary derivative, the equation would no longer be covariant.

An analogy comes from finance, calculating the difference in the prices of a commodity in two different countries. You can’t just subtract the price of one from that of the other, you must take into account the difference in currencies, the exchange rate. If I want the difference in price in something between France and the USA in euros, I must first change the USA price in American dollars to euros by using the exchange rate, and then take the difference. Tho exchange rate is therefore a kind of connection.[16]

We are interested in matter particles, so we consider one, two or three fermions. (We’ll see why this choice in a moment.) In the latter two cases, we take them to have equal masses in order to make the equations easier (possible) to solve. Doing a local transformation adds a term or terms to each equation. The extra terms come from taking the derivative of the transformation operating on the free-particle field.[17] This gives the usual term for the field plus one or more additional terms due to the transformation, the number depending on the dimension of the group, which is just the number of particles. In the case of a local transformation, the transformation is constant, so its derivative (its rate of change) is zero and gives us no problem. But in order for our equations to be covariant in the case of a local transformation, we must finagle them so as to cancel the extra terms from the

15 Schwartz, Matthew D, *Quantum field theory and the standard model*, 20.

16 Schwichtenberg, Physics from finance. Maldacena, The symmetry and simplicity of the laws of physics and the Higgs boson. On-line.at arxiv.org/abs/1410.6753.

17 Using something called differentiation by parts (the product rule), a way of taking a derivative when more than one part of its object can change.

derivative of the transformation. This is why we use the connection term in the covariant derivative – to cancel out the extra term due to the local transformation.

Let's consider first the relatively simple case for a single fermion, such as an electron. Its equation does not stay the same under local transformations, but picks up an extra term. In order to deal with this, we change our derivative to a covariant derivative with an added connection which is a product of constant terms and a 4-vector field. Consistency of the equations tells us how the field must transform and – lo, behold – it is just like the vector potential of classical EM. In fact, it *is* the vector potential.

The result is a striking success for three reasons.

- The connection field turns out to be the EM vector potential used to calculate the electric and magnetic fields.
- Using the covariant derivative, the equation is now invariant under the transformation.
- If we write out the covariant derivative, the extra term is a product of the electron (fermion) and vector fields, so it represents the interaction between the two. We can now calculate the interaction between an electron and a photon. Wow!

The particle form of light, the photon is a massless vector boson (spin 1), and from the Proca equation we also know how that transforms under the same complex (local) rotation. Since we want the interaction of the electron with a photon, we need to include the photon too. Whew! So now we have the electron plus the photon plus the product (interaction) term.

(electron) + (photon) + (interaction).

And when we look at the result of the local transformation of this equation, we see that it is conserved unchanged. And it's the same ***gauge invariance*** found in Maxwell's equations of EM.

Once more, with feeling. Rather than adding the extra interaction term to the equation, we can include it by redefining the derivative in the Dirac equation. This is logical, since the derivative depends on how the system changes from one point in spacetime to the next. This ***covariant derivative*** now includes an extra term which takes into account the change in the transformation from one point to the other. Since the additional term serves to connect the two points, it is called a ***connection***. This is generally considered to be a more satisfying reasoning for inclusion of the extra term. It Is the one assumed when giving the usual explanation of gauge invariance.

How do we interpret these results? Well, we have already said that electrons are fermions are matter particles, and that photons are bosons are force-carrying particles. The photon was necessary in order to guarantee gauge invariance of the free-electron equation and so the photon is called a ***gauge particle***.

The same method applies to a group of two or three fermions, and we have the following cases:

- In the single-fermion case, the single force-carrying gauge particle is the massless photon of EM.
- In the two-fermion case, there are three force-carrying gauge particles, which are interpreted to be the $W^{\pm}$ and the Z^0 particles of the weak interactions. Alas, experiments show that these three particles are very far from being massless or of equal mass. But don't worry, we have a plan, to be divulged in the next section.
- In the case of three fermions, interpreted to be the three ***quarks*** in a proton or neutron, there are eight force-carrying gauge particles, which are the massless ***gluons*** which carry the strong interaction.

The difference in the number of gauge particles comes from the dimension of the unitary group describing the initial n-fermion case, U(1), SU(2) or SU(3), the number of gauge particles being $n^2 - 1$ for the special (SU) groups..

Hey, isn't that great! All particle interactions are tied together by the requirements of symmetries, which give us the interaction terms between different particles. And the very existence of the gauge bosons comes out of the same calculations.

6.3. How about mass? The Higgs mechanism

We're still stuck with that problem with the masses of the W and Z bosons, which come out as 0 from the equations taking into account invariance under a local transformation.

First, several words about potential energy. (Skip this if you already know about it.) These days, physicists don't talk so much about forces, more about energy. The two notions are equivalent in their results. Indeed, most forces can be derived from a potential energy. Imagine a potential energy whose value on a graph looks like a U centered around x=0 and with the y axis being the value of the potential energy. Thermodynamics tells us that systems like to be in a state of minimum energy, so at the bottom of the U. That's in terms of energy, but we can also see it in terms of forces. The force is the negative of the rate of change (yes, the derivative) of the potential. So in the middle of the U, where the potential-energy curve is approximately flat, the derivative is zero and there is no force pushing the system away from that point. Go to the right some, though, to where the curve is turning up in the U. Now the curve is steeper so it's negative derivative is pushing the system back toward zero. Farther to the right, the curve is steeper, so the force pushing the system back toward zero is stronger yet. So now we can talk about potential energy curves, not forces, knowing that our system wants to be at the minimum value of the potential-energy curve.

Meanwhile, back with the W and Z bosons, the trick is to accept the math and assume that these particles *did* have zero mass – just after the Big Bang (in the first 10^{-12} seconds or so of the Universe) – but then something happened to spoil this. In the late 1960s, Weinberg, Salam and Glashow came up with the electroweak theory of EM and weak interactions. Their assumption is that during those first bits of a second, not only the W and Z particles but also the electron had zero mass, and that the EM and weak interactions were therefore symmetric and were really the same. In 1964 a trick was proposed, originally thought of by Higgs, Englert and at least four other physicists.[18] A form of potential energy field was proposed which looks not like a U, but like a sombrero or the bottom of a wine bottle[19] and has its minimum energy value at a non-zero value of the field. This was done simply by adding in an extra term to the potential. You can show this on a graph where the vertical z-axis is the energy of the field and the x and y axes (or r ant θ, if you prefer) are the value of the field. Usually, zero is zero and gravitational potential energy in classical mechanics, for instance, has its minimum energVanessa Kirbyy at zero of the field (taken to be the surface of the Earth). But the Higgs field can have its minimum-energy value out a way from zero at a non-zero value of the field, like the circular lowest part of the wine bottle (or hat). Saw off all but the bottom couple centimeters of a wine bottle and pose a marble on the high spot in the middle, corresponding to zero of the field. It is obviously not stable. The marble will roll down into the circular trough a centimeter or two from the center. This is the minimum-energy value and it is not at the center of the field represented by the bottom of the bottle (or hat). This is what we suppose happened as the Universe was around 10^{-12} seconds "old". At that time, the Universe (the marble) slipped into a particular but arbitrary place in the trough and lost its symmetry. Physicists call this process ***spontaneous symmetry breaking***.

If we put this non-zero, but minimal, field value into the equations and fiddle around a lot (moving the coordinate system from the center out to the point in the trough, using a covariant derivative to introduce a gauge transformation plus a bit more), we wind up with W and Z particles' and electron's all having mass. We also have a scalar (spin zero) particle with mass and this is the famous (notorious?) Higgs particle.

This is because they all interact with the Higgs field, which also comes out of the calculation. To repeat: The gain of mass of a particle results from its interaction with the Higgs field. The equations show no resulting interaction between the Higgs field and the photon, which therefore remains massless, as it is known to be.

The calculation defends the notion of a combined electroweak force, at least just after the Big Bang.

It predicts the existence of the Higgs field and particle. The discovery of a candidate for the Higgs particle was announced at CERN on 4 July 2012 and its identity as the Higgs was confirmed within a year..

6.4. Path integral method and Feynman diagrams

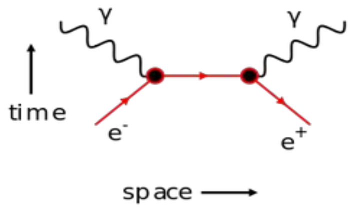
Now we have many particles, thanks to second quantization, and interactions due to gauge bosons, thanks to

18 I don't know why only Higgs and Englert were selected for the Nobel Prize. Nobody accuses the prize committee of fairness – or courage.

19 Usually called, I know not why, a "Mexican hat" potential. I prefer wine bottles.

gauge invariance. We still need a way to calculate what happens when two particles meet.

The probability of a particle's going from here to there depends on the particle, on here, on there and on what happens in between, i.e., on the path taken from here to there. Feynman's *coup de génie*, in 1948, was to realize that one should sum up the probability amplitudes (probability = square of amplitude) for all possible paths from here to there, and that many of them give a negligible contribution to the total. But there was more. He showed how the contribution from each path could be represented by a diagram. The diagrams are now named after him. Here is an example, for and electron-positron annihilation into two photons.



Feynman diagram for electron-positron (e^- - e^+) annihilation to two photons γ, by bitwise via Wikipedia.[20]

Calculating the "amplitude" for this (i. e., a wave function before squaring it to get a probability) is not trivial, but Feynman broke the problem down into manageable parts. The amplitude is the product of terms: one for each incoming or outgoing particle, one for each vertex and one for what happens in between vertices. It's the in-between part that is complicated, but that can be calculated by starting from the solutions to the free-particle equations for the particles and doing a lot of math. Once the formula is calculated for a particular interaction, the formula can be reused over and over again in similar diagrams. It's almost copy-paste.

The problem is that the results tend to be infinite, but even this can be resolved by a horrendously complicated subject called ***renormalization***, into which we will very definitely not go!

7. Summing up

In a minimum of words, we have found the following points.

- SR requires the speed of light to be constant as seen from an inertial frame. This in turn requires us to use 4-d spacetime
- Second quantization permits the creation and annihilation of multiple identical particles
- Symmetry requires the use of connections based on gauge fields and these are the forces of particle interactions. It is also the basis of conservation laws
- The Higgs mechanism shows how particles acquire mass through spontaneous symmetry breaking.
- QFT: It's all fields.

Except for SR, these are all quantum mechanics or applications thereof.

8. Equations of quantum electrodynamics (QED)

You can skip this, but I can't resist these equations showing how QED comes about from symmetry considerations. And if even you don't understand all the math, looking at them piece by piece will aid in understanding what this gauge-field business is all about. If not, you can pat yourself on the back and go get a cup of coffee.

First, let's admire an equation, the beautiful (yes!) Dirac Lagrangian for a free (non-interacting) spin-½ particle like an electron.

$$\mathcal{L} = \bar{\Psi}(i\gamma^{\mu}\partial_{\mu} - m)\Psi. \tag{8.1}$$

20 https://commons.wikimedia.org/wiki/File:Feynman_EP_Annihilation.svg

Here, Ψ is the electron wave function and $\bar{\Psi}$ is its Hermitian adjoint.[21] What that means, for our purposes, is that the total (integrated over space) $\bar{\Psi}\Psi$ is equal to one, the probability that the particle is somewhere or anywhere.

What do we see in this equation? The simplest thing is the term with m, and since m is a constant, this gives us

$$\bar{\Psi}m\Psi = m\bar{\Psi}\Psi = m,$$

the mass of the electron. Actually, we give it the mass, which we have determined by experiment elsewhere.

The other term is more interesting and has two parts. One is that Greek thingie, the γ^μ. Dirac included this because he needed the equation to be consistent with the relativistic energy-momentum relation (the dispersion relation, in physics speak)

$$E^2 = m^2 + p^2.$$

Note that if the electron is sitting still, its momentum p is zero and this equation reduces to Einstein's famous $E = mc^2$. (You may not recognize it because in the other equations, we have set the speed of light, c, and the Planck constant, $\bar{h}$, to 1, a trick to simplify the equations.)

In SR, a Greek subscript or superscript like μ indicates a value of 0 to 3, corresponding to time and the three perpendicular (orthogonal, in math speak) spatial directions. So there are four quantities γ^μ, which I present for your contemplation and enjoyment.[22]

$$\gamma^0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$
$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

These are therefore 4x4 matrices which multiply the 4-d partial derivative ∂_μ. That's because of the μ super and subscripts Why all this? Because electrons have a spin which – remember – exists in its own space, spinor space, and these are the operator part handling the spin.

Now for the third part. The derivative term ∂_μ expresses the rate of change in the system in time (μ = 0) and space (μ = 1,2,3). If the wave function Ψ is transformed by a constant operator, as is the case with a global transformation, everything looks the same afterwards in their relative positions. If, however, the transformation is not constant, then the change here will be different from what it is over there and this fact must be taken into account in calculating the change in Ψ. This line of reasoning requires a change in ∂_μ which will lead us to the covariant derivative, D_μ, presented in section 6.2.

Here's how the math goes. We consider the local U(1) unitary transformation

$$U = e^{i\alpha(x)},$$

which is a phase rotation through an angle which varies from point to point in space. Under this transformation, the Dirac Lagrangian changes as follows:

$$\mathcal{L}_{Dirac} = \bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi \rightarrow \mathcal{L}_{Dirac} - \partial_\mu\alpha\bar{\Psi}\gamma^\mu\Psi, \tag{8.2}$$

because of the change due only to the transformation. What we then do is, we write a function which translates the field there into the field here. We can use this to write a derivative in which the two terms are coherent and we can do the subtraction. But that is only done at the cost of an extra term in the derivative. Adding this term makes the Dirac equation invariant under a local U(1) transformation. The derivative with its new term is now the

21 The Hermitian adjoint matrix is inverted complex conjugate of the original matrix. Don't worry about it.

22 These are in the co-called chiral basis. Don't worry if that means nothing to you.

covariant derivative,

$$D_\mu\psi(x) = \partial_\mu\psi(x) + iqA_\mu\psi(x). \tag{8.3}$$

Consistency[23] then requires that the A_μ transform as

$$A_\mu(x) \to A_\mu(x) - \frac{1}{q}\partial_\mu\alpha(x), \tag{8.4}$$

which is the transformation of the Proca (spin = 1) equation. The result is an added term to the Dirac equation,

$$-q\bar{\psi}\gamma^\mu A_\mu\psi, \tag{8.5}$$

which is a product of the two types fields, electron and photon, and is just the interaction term we were looking for.

Now we must consider the photon with which we want this electron to interact. Since a photon has spin 1 and zero mass, we use for it the m=0 Proca equation,

$$\mathcal{L}_{Proca} = -\frac{1}{2}(\partial^\mu A^\nu\partial_\mu A_\nu - \partial^\mu A^\nu\partial_\nu A_\mu) \tag{8.6}$$

which stays the same (is invariant) if the local $U(1)$ transformation if A_μ transforms like (8.3), which it does! For any $\alpha(x)$ this is a representation of the same $U(1)$ group for spin equal to 1. It also is the gauge transformation of the standard EM vector potential from classical EM and Maxwell's equations.

So here is the Lagrangian for QED, quantum electrodynamics:

$$\mathcal{L}_{Dirac+int+Proca} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi + qA_\mu\bar{\psi}\gamma^\mu\psi - \frac{1}{2}(\partial^\mu A^\nu\partial_\mu A_\nu - \partial^\mu A^\nu\partial_\nu A_\mu).$$

Including the gauge field to form the ***covariant derivative***

$$D_\mu = \partial_\mu + iqA_\mu \tag{8.7}$$

"simplifies" this to looking pretty much like a sum of the Dirac and Proca equations (8.1) and (8.6).

$$\mathcal{L}_{Dirac+int+Proca} = \bar{\Psi}(i\gamma_\mu D^\mu - m)\Psi - \frac{1}{2}(\partial^\mu A^\nu\partial_\mu A_\nu - \partial^\mu A^\nu\partial_\nu A_\mu).$$

So in this sense, equation (8.7) for the covariant derivative tells us that it is the EM vector field A_μ itself which connects one point to the next. At the same time, the field is the photon and gives us the interaction term for the electron and photon. All by insisting on covariance under a local U(1) transformation. This is a big deal indeed. Remember from second quantization that in fact the photon is an excitation of the field A_μ.

We can do similar calculations for two or three fermions under local SU(2) or SU(3) transformations and will find the results already mentioned in the bulleted list of section 6.2. In each case, the covariant derivative has added to it the fields of which the force-carrying particles are excitations. This gives us interaction term we need. Poof!

Thoughts:

- The non-local gauge transformation requires inclusion of the interaction which adds a connection term which is none other than the field in the interaction.[24]
- Turning this around, we can say that *a force (interaction) is due to a non-local gauge field which connects one point to the next*. (It must be understood that we mean infinitesimally separated points connected by a translation, rotation or Lorentz transformation.) This would seem to say that the non-local field is the force, in the sense that the force-carrying particles are excitations of this field. It is related to the transformation through its change under the transformation, equation (8.4).

23 I'm not saying consistency with what because the what, because it's the translation function, which I haven't written. For details, see my paper on "Symmetry, groups and quantum field theory".

24 Ok, it's multiplied by some constants, but they don't change.

If you are not saturated now, consider one more thing. It is the custom to factor out a coupling constant q like we have done in (8.5). Then the Noether current

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi_i)} \delta\Psi = -q\bar{\Psi}\gamma^\mu\Psi,$$

is the ***electric four-current***. The zeroth component of this is the electric charge density, so the total charge is the integral of this quantity:

$$Q = \int d^3x J^0 = -q \int d^3x \bar{\Psi}\gamma^0\Psi = -q,$$

because of normalization. So by Noether's theorem, global U(1) symmetry means electric charge is conserved. Similar results are found for momentum, energy, spin, isospin and more.