Summary of spin

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1. In words

A rotation is a transformation acting on an operator or a state vector (wave function). We know from our 3-d world that physical systems are symmetric under rotations, meaning the object rotated still looks the same as before. This means that the generator of rotations, angular momentum, is conserved. Lie group theory tells us that real rotations are representations of SO(3) and complex ones of SU(2).[1]

Experiment (Stern-Gerlach) shows that electrons have a magnetic moment which acts like that of an object with angular momentum of only two distinct values along the field direction. Since it acts like angular momentum, we call it ***spin***.

We measure the spin component along one axis, usually taken to be the z-axis, to be either $\frac{\hbar}{2}$ or $-\frac{\hbar}{2}$. So according to the rules of angular momentum eigenvalues (J^2, J_z), the spin space must be two-dimensional, just as if the measured values were -1, 0 or +1, it would be three-dimensional. The lowest-dimension representation for which the commutation relations of angular momentum hold is a 2-d representation of SU(2). We know it is of order 3 $(n^2 - 1)$, as is SO(3) $(\frac{1}{2}n(n-1))$. This corresponds to the parameters, three rotation angles, which is also the dimension of the space we work in, good old 3-d space.

From the results of using differently oriented S-G apparatuses in series, we can deduce the operators for angular momentum in the three directions of space. It lives in 3-d space because that is where we are when we measure it. The expectation value of the vector defined by the spin operators, S_x, S_y, S_z, rotates like any other vector. But the eigenvectors of SU(2) have a foot in 3-d (since there are three of them) and in 2-d space (since they are composed of 2-d matrices). Relativistic QM (QFT) will tell us this is necessary in order to maintain Lorentz invariance. So we have objects in 3-d space which transform like vectors, but which include a component which is expressed (represented) in 2-d eigenspace, which we call spin space, or ***spinor*** space. In a somewhat imprecise way, we can consider the 2-d eigenspace to be "linked" to the 3-d observable space through the state vector and its expression in terms of Pauli matrices By convention, we consider spinor space to be an "internal" space. But internal to what…?

A general rotation through angle θ about a unit vector $\hat{n}$ is represented by

$$exp(i\theta\hat{n} \cdot \vec{J}/\hbar), \tag{1.1}$$

with $\vec{J}$, the generator of the rotation, being the angular momentum. In the case of *spin space*, the AM operator is $\vec{S} = \frac{1}{2}\hbar\vec{\sigma}$, the elements $\vec{\sigma}$ being the Pauli matrices. In this case, the rotation operator looks like

$$D_n(\theta) = exp(i\theta\hat{n} \cdot \vec{S}/\hbar) \rightarrow exp(i\left(\frac{\theta}{2}\right)\hat{n} \cdot \sigma) \tag{1.2}$$

and the effect on the 2-d spin part of the state vector is that of half the angle of rotation. This is because the rotation operator has S_z and friends as generators and these return a value of $S_z = \frac{\hbar}{2}$. The ½ comes out in the factor showing the effect of the rotation on the measured spin.[2] In 3-d space, the expectation value of the spin still rotates like a vector through 360°.

In the case of a rotation by 2π, the rotation is indeed through 360°, but the effect on the state vector of a spinor is to multiply it by -1. This operation changes the phase of the state vector, which has no effect on expectation values. Rotation leaves observables alone. Experiments using neutron interferometers have validated this effect.[3]

1 More on Lie groups in my notes on symmetry and QFT.
2 Sakurai and Napolitano, 158-159.
3 Sakurai and Napolitano, 158-159.

2. In math

Let's start with experiment. The method and most equations are borrowed from chapter 3 of Sakurai and Napolitano.

A ***Stern-Gerlach*** apparatus shows that electrons (or nuclei with one free electron) behave in a non-uniform magnetic field as it they had magnetic moments of only two values, oriented either up or down along the field. Since the magnetic moment is proportional to the ***angular momentum*** of the electron, it seems this too can only take on two values. We take the direction of the magnetic field to define the z-axis. Selecting, say, the emerging up component by blocking the down one, we then can send this beam of pure z-up through another S-G apparatus and measure a perpendicular component, say, in the x direction. We see an equal mixture of components along the x-axis, suggesting that the z-component is a mixture of x, say, x left and x right in equal proportions. Any pure beam, x up or x left or y in, seems to be a superposition in any orthogonal direction..

If we indeed take these results to be indicative of an angular momentum of two eigenvalues, it must be described by the Lie algebra of SU(2), so we start by supposing up and down *states* along the z-axis, represented by Dirac kets.

$$|u\rangle \text{ or } |+\rangle \quad \text{and} \quad |d\rangle \text{ or } |+\rangle \qquad (2.1)$$

with the z-spin ***operator*** having eigenvalues modulo $\frac{\hbar}{2}$, where the ½ is the spin value and the reason for $\hbar$ will become clear later,

$$S_z|\pm\rangle = \pm\tfrac{\hbar}{2}|\pm\rangle \qquad \text{(2.2, S-1.91)}[4]$$

Note well: S_z is the ***operator*** for spin in the z direction; $|\pm\rangle$ is the Dirac-style ***state vector*** for the z-component of the spin to be up or down. The ***eigenvectors*** can be represented as 2-d matrices

$$\chi_+ \equiv |+\rangle \doteq \begin{pmatrix}1\\0\end{pmatrix} = |S_z+\rangle \quad \text{and}$$

$$\chi_- \equiv |-\rangle \doteq \begin{pmatrix}0\\1\end{pmatrix} = |S_z-\rangle. \qquad \text{(2.3, S-1.93a)}$$

where $\doteq$ means "is represented by". Then a general spin state can be written

$$|\alpha\rangle = |+\rangle\langle+|\alpha\rangle + |-\rangle\langle-|\alpha\rangle \doteq \begin{pmatrix}\langle+|\alpha\rangle\\ \langle-|\alpha\rangle\end{pmatrix} = \chi \qquad \text{(2.4, S-3.45a)}$$

and, similarly,

$$\langle\alpha| = \langle\alpha|+\rangle\langle+| + \langle\alpha|-\rangle\langle-| \doteq (\langle\alpha|+\rangle, \langle\alpha|-\rangle) = \chi^\dagger. \qquad \text{(2.5, S-3.45b)}$$

The state χ and it adjoint $\chi^\dagger$ are ***spinors***.

Any operator may be represented by its ***spectral representation***, the sum of its eigenvalues multiplied by the projection operator for that eigenstate, which in this case is

$$S_z = \tfrac{\hbar}{2}(|+\rangle\langle+| - |-\rangle\langle-|). \qquad \text{(2.6, S-1.90)}$$

By (2.3), this gives the z operator

$$S_z = \tfrac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \qquad \text{(2.7, S-1.93b)}$$

S-G then tells us that the x component must be a mixture of z-up and z-down:

$$|S_x\pm\rangle = \tfrac{1}{\sqrt{2}}(|+\rangle \pm e^{i\delta_1}|-\rangle). \qquad \text{(2.8, S-1.102)}$$

Then substitution into the spectral representation of the x-spin operator becomes

4 Equation numbers S-nnn are the numbers from Sakurai and Napolitano.

$$S_x = \frac{\hbar}{2}[e^{-i\delta_1}(|+\rangle\langle-|) + e^{i\delta_1}(|-\rangle\langle+|)] \quad \text{(2.9, S-1.104)}$$

Making a similar representation for S_y and considering the equal distribution of measure of y-spin after selection on x, we find for the *states*:

$$|S_x\pm\rangle = \frac{1}{\sqrt{2}}(|+\rangle \pm |-\rangle) \quad \text{(2.10, S-1.110a)}$$

$$|S_y\pm\rangle = \frac{1}{\sqrt{2}}(|+\rangle \pm i|-\rangle) \quad \text{(2.11, S-1.110b)}$$

$$|S_x\pm\rangle = \frac{1}{\sqrt{2}}(|+\rangle \pm e^{i\delta_1}|-\rangle) \quad \text{(2.12, S-1.102)}$$

and, from (2.7) and its y analog, for the *operators*

$$S_x = \frac{\hbar}{2}[(|+\rangle\langle-|) + (|-\rangle\langle+|)] \quad \text{(2.13, S-1.111a, 3.25)}$$

$$S_y = \frac{i\hbar}{2}[-(|+\rangle\langle-|) + (|-\rangle\langle+|)] \quad \text{(2.14, S-1.111b, 3.25)}$$

$$S_z = \frac{\hbar}{2}(|+\rangle\langle+| - |-\rangle\langle-|) \quad \text{(2.15, S-1.90 3.25)}$$

From these follow the ***fundamental commutation*** and ***anticommutation relations***

$$[S_i, S_j] = i\epsilon_{ijk}S_k \quad \text{and} \quad \{S_i, S_j\} = \frac{1}{2}\hbar^2\delta_{ij}, \quad \text{(2.16, S-1.113, 114)}$$

$$S^2 = S_x^2 + S_y^2 + S_z^2 \quad \text{and} \quad [s^2, S_i] = 0. \quad \text{(2.17, S=1.116, 1,118)}$$

Since a general angular momentum J_i is the generator of rotations $D_n(\phi)$,

$$\langle J_x\rangle \rightarrow_R \langle\alpha|J_x|\alpha\rangle_R = \langle\alpha|D_z^\dagger(\phi)J_xD_z(\phi)|\alpha\rangle$$

$$\rightarrow exp(\frac{iJ_z\phi}{\hbar}\Big)J_xexp\Big(\frac{-iJ_z\phi}{\hbar}\Big) = \text{...series}$$

and finally

$$\langle J_x\rangle = \langle J_x\rangle cos\phi - \langle J_y\rangle sin\phi \quad \text{(2.18, S-3.24)}$$

For the spin expectation values

$$\langle S_x\rangle \rightarrow_R \langle\alpha|S_x|\alpha\rangle_R = \langle S_x\rangle cos\phi - \langle S_y\rangle sin\phi \quad \text{(2.19, S-3.26)}$$

$$\langle S_y\rangle \rightarrow \langle S_y\rangle cos\phi + \langle S_x\rangle sin\phi \quad \text{(2.20, S-3.27)}$$

$$\langle S_z\rangle \rightarrow \langle S_z\rangle \quad \text{(2.21, S-3.28)}$$

So, in analogy with (2.18), the expectation value of the spin operator under rotation behaves exactly as would a classical vector. Then for a general state vector,

$$|\alpha\rangle = |+\rangle\langle+|\alpha\rangle + |-\rangle\langle-|\alpha\rangle, \quad \text{(2.22, S-3.31)}$$

rotation about the z-axis is given by

$$D_z(\phi)|\alpha\rangle = exp(\tfrac{-iS_z\phi}{\hbar})|\alpha\rangle = e^{-i\phi/2}|+\rangle\langle+|\alpha\rangle + e^{i\phi/2}|-\rangle\langle-|\alpha\rangle. \quad \text{(2.23, S-3.32)}$$

That factor of ½ is important, because for a rotation of the system by angle 2π, it means that

$$\phi = 2\pi \Rightarrow \; D_z(2\pi)|\alpha\rangle = -|\alpha\rangle, \quad \text{(2.24, S-3.33)}$$

the state picks up a minus sign. We must rotate through 4π to get back to the original state. From group theory, we know this is a result of SU(2) being the double cover of SO(3).[5] Fortunately, the expectation value is unfazed by this change of phase.

$$\langle S_z\rangle \rightarrow_R \langle\alpha|S|\alpha\rangle_R \rightarrow (-1)^2\langle S_z\rangle \quad \text{(2.25)}$$

Now we identify the spin matrix elements in terms of 2x2 matrices:

5 See my document on symmetries and QFT.

$$\langle \pm|S_k|+\rangle = \left(\frac{\hbar}{2}\right)(\sigma_k)_{\pm,+} \quad \text{and} \quad \langle \pm|S_k|-\rangle = \left(\frac{\hbar}{2}\right)(\sigma_k)_{\pm,-} \qquad \text{(2.26, S-3.48)}$$

Remembering the spinors of (2.4) and (2.5), the expectation value of a spin component can be written

$$\langle S_k\rangle = \langle\alpha|S_k|\alpha\rangle = \sum_{a'=+,-}\sum_{a''=+,-}\langle\alpha|a'\rangle\langle a'|S_k|a''\rangle\langle a''|\alpha\rangle$$
$$= \frac{\hbar}{2}\chi^\dagger\sigma_k\chi. \qquad \text{(2.27, S-3.49)}$$

Using (2.13-2.15) and (2.26), we can derive the Pauli matrices

$$\sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \sigma_y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \quad \sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \qquad \text{(2.28, S-3.50)}$$

Alternatively, we could have derived the equations for the x, y and z components of of the operator by using $|+\rangle$ and $|-\rangle$ as expressed in (2.3) and putting them into (2.13 through 2.13).

Their properties include

$$\sigma^2 = 1, \qquad \{\sigma_i, \sigma_j\} = 2\delta_{ij} \qquad [\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k, \qquad \text{(2.29, S-3.51a, 3.52, 3.53)}$$
$$(\sigma\cdot\vec{a})^2 = |a|^2, \qquad \text{(2.30, S-3.59)}$$
$$(\sigma\cdot\hat{n})^2 = \begin{cases} 1, & n \text{ even}\\ \sigma\cdot\hat{n}, & n \text{ odd}\end{cases}. \qquad \text{(2.31, S-3.61)}$$

Now we can consider the rotation *operator* in its 2x2 matrix representation.

$$D_n(\phi) = exp\left(\frac{-iS\cdot\hat{n}\phi}{\hbar}\right) = exp\left(\frac{-i\sigma\cdot\hat{n}}{2}\right). \qquad \text{(2.32, S-3.60)}$$

Using a Taylor expansion separated in two parts and (2.31) , this becomes simpler:

$$D_n(\phi) = exp\left(\frac{-i\sigma\cdot\hat{n}}{2}\right) = \mathbf{1}cos\left(\frac{\phi}{2}\right) - i\sigma\cdot\hat{n}sin\left(\frac{\phi}{2}\right). \qquad \text{(2.33, S-3.62)}$$

As in (2.24), rotation through 2π results in $D_n(2\pi) = -1$. Putting in the Pauli matrices gives

$$D_n(\phi) = exp\left(\frac{-i\sigma\cdot\hat{n}}{2}\right) = \begin{pmatrix} cos(\frac{\phi}{2}) - in_z sin(\frac{\phi}{2}) & (-in_x - in_y)sin(\frac{\phi}{2})\\ (-in_x + in_y)sin(\frac{\phi}{2}) & cos(\frac{\phi}{2}) + in_z sin(\frac{\phi}{2})\end{pmatrix}. \qquad \text{(2.34, S-3.63)}$$

Not $\vec{\sigma}$, the vector of Pauli matrices, but $\chi^\dagger\vec{\sigma}\chi$, with χ the spinor state vector of (2.4) transforms like a vector. For instance,

$$D_3(\phi)(\chi^\dagger\sigma_1\chi) = exp\left(\frac{i\sigma_3\phi)}{2}\right)\sigma_1 exp\left(\frac{-i\sigma_3\phi}{2}\right) = \sigma_1 cos\phi - \sigma_2 sin\phi, \qquad \text{(2.35, S-3.66)}$$

the 2x2 analog of (2.18).