

# The story of hbar

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The two major physics breakthroughs (even, revolutions) of the 20<sup>th</sup> century may be said to be symbolized by two quantities,  $c$ , the speed of light in a vacuum, and  $\hbar$ , Planck's constant  $h$  divided by  $2\pi$ .

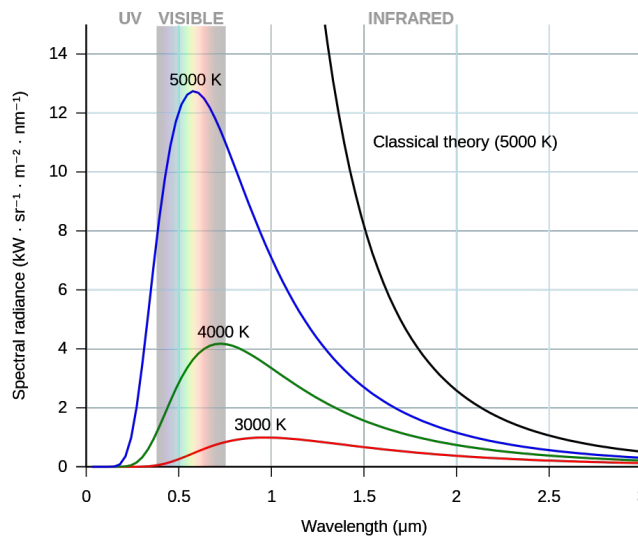
Light speed is fairly simple to understand. It's enough to measure it in a vacuum and find that it not only has the same value everywhere but that its value is exactly (well, whatever that means in physics...) what is predicted by Maxwell's equations --

$$c = \frac{1}{\mu_0 \epsilon_0} = 3 \times 10^8 \text{ m/s.} \quad (1)$$

The birth of  $h$  was more painful, even contrary to its mother's wishes, and (s)he never was quite happy with it.

## 1. Planck and his black bodies

It all starts with black-body radiation. Wikipedia says: "A perfectly insulated enclosure which is in thermal equilibrium internally contains blackbody radiation, and will emit it through a hole made in its wall, provided the hole is small enough to have a negligible effect upon the equilibrium." That's all we need to know. In the late 19<sup>th</sup> century, the frequency distribution of BB radiation had been measured and gave the following curves of radiation density versus wavelength for different temperatures, each at a different temperature.<sup>1</sup> (The curves are fits; data not show.)



Numerous scientists had studied this phenomenon and tried to find a "law" – an equation – to fit the data. There was a sort of war between two groups. On the one side were the so-called atomists, like Boltzmann, who were advocates of the statistical-mechanical approach based on probabilities, according to which entropy could decrease, just only rarely. Planck was a member of the opposite side, those who put all their faith in the second law of thermodynamics, according to which the entropy of the universe can never decrease.

In 1896, Wilhelm Wien came up with an equation to describe energy density of BB radiation. In its form at the time, it looked like this.

$$\rho(\nu, T) = a\nu^3 \exp(-b\nu/T).$$

Because it fitted the data of the time, it was fairly widely accepted --- until new data showed it was wrong at

<sup>1</sup> Wikipedia, black body svg. [https://en.wikipedia.org/wiki/File:Black\\_body.svg](https://en.wikipedia.org/wiki/File:Black_body.svg).

lower energies. Another proposal, from Rayleigh corrected by Jeans, was, therefore, the Rayleigh-Jeans law:

$$\rho(\nu, T) = 8\pi(\nu^2/c^3)kT,$$

where  $k$  is the Boltzmann constant from that scientist's equation for entropy:

$$S = k \log(W). \quad (2)$$

It was even worse, predicting infinite total energy in the ultraviolet (low wavelength) region. It is shown as the classical law in the figure.

Planck, much against his own wishes, was obliged to take the statistical version into account. He deduced the equation rather easily, but explaining it was more complicated. The law is, in its original version of 1900,

$$u(\nu, T) = \frac{a\nu^3/c^3}{\exp(b\nu/T) - 1},$$

which approximates to both the Wien and Rayleigh-Jeans in the appropriate ranges. A later version changed the constants involved:

$$u(\nu, T) = \frac{8\pi\nu^2}{c^3} \frac{h\nu}{\exp(h\nu/kT) - 1}, \quad (3)$$

where, in particular, constant  $b$  is replaced by  $h/k$ . Planck picked  $h$  because he referred to it as a *Hilfsgrösse*, or *auxiliary term*.

The problem was that, even though he now had the equation, based on the data, he did not know why it worked or how it fit into the atomist-thermodynamic disagreement. The results of his efforts would surprise even him.

Everyone agreed that the energy must be in the form of some tiny oscillators, be they real or what we would now call virtual. Planck started by studying a model system composed of electromagnetic waves interacting with damped oscillators. He first derived a general equation in terms of  $U(\nu, T)$ , the average internal energy of the oscillators. Note that  $U$  depends only on the frequency and intensity (temperature) of the oscillators.

$$\rho(\nu, T) = \frac{8\pi\nu^2}{c^3} U(\nu, T). \quad (4)$$

After much intellectual labor, Planck managed to derive the oscillator entropy in both the thermodynamic and statistical contexts. For the derivation based on thermodynamics, he used the entropy equation  $dS = dU/T$  and  $U$  from his equations (3) and (4) to develop an equation for the entropy. For the statistical approach, Boltzmann had proceeded by using a model based on a system of imagined energy states and assigning atoms to them statistically. Planck took the opposite approach, assigning *indistinguishable energy elements* to the atoms. He worked out the probabilities and used the result for  $W$  in Boltzmann's own equation (2) to come up with an equation for the entropy of the same form as the one from thermodynamic principles. Comparison of the two showed them to be equivalent – indeed, identical – except for the terms  $E$  in the one and  $h\nu$  in the other. This could only mean that

$$E = h\nu. \quad (5)$$

Because the  $E$  in this expression is the energy of each individual energy element (oscillator) in the statistical derivation of the equation, it can only mean that each element possesses a fixed value or *quantum* of energy as given by the equation. This equation has changed the world. *Danke schön, Herr Planck.*

High point: Light, previously firmly established by Maxwell's equations to be a wave, has particle aspects, namely, energy quanta.

## 2. Einstein and the photo-electric effect

Notwithstanding Planck's success with BB radiation, the notion of quanta still did not catch on, not even with Planck himself. Then Einstein arrived on the scene to explain BB radiation as actual lumps of light, i.e., discrete quanta of radiation. He used this idea in his explanation of the photoelectric effect. That was in 1905, the same year when he published papers on Brownian motion, Special Relativity and the equivalence of mass and energy.

No wonder it is called his *annus mirabilis*. Einstein realized that light quanta of energy  $h\nu$  could explain why low-frequency light does not eject electrons from atoms -- because they do not have the required energy. Increasing the number increases number of quanta but not their energy.

Even after this, the notion of quanta was not exactly mainstream. Then in 1914, Robert Millikan ran an experiment which demonstrated the quantum effect on photoelectricity. In 1914, he verified Einstein's predictions and obtained an accurate measurement of the Planck constant. Einstein got the Nobel Prize for his work on the photo-electric effect in 1921 and Millikan, in 1923. The name of the electromagnetic quantum, *photon*, was coined by Gilbert Lewis in 1926.

High point: EM radiation, or light, exists, at least in some cases, as particles, called photons. Wait a minute, though. It's also a wave...

### 3. Bohr and energy levels of atomic electrons

Heated hydrogen emits a series of colors at different frequencies. An experimentally derived formula for the wavelengths of one such series was proposed in 1885 by Jakob Balmer. In 1888, Johannes Rydberg published a more general formula, which included Balmer's. The wavelengths were given by

$$\frac{1}{\lambda} = R_H \left( \frac{1}{n_1^2} - \frac{1}{n_2^2} \right),$$

where  $n_1$  and  $n_2$  are integers with  $n_2 > n_1$ . The Balmer series was the Rydberg series with  $n_2=2$ . Between 1911 and 1918, this result was explained by Neils Bohr, when he developed his model of the hydrogen atom as an electron in orbit around a proton. He based the model on classical mechanics, Coulomb's law of electric force and two quantum assumptions:

- The orbital energy of an electron is given as an integral multiple of Planck's constant times half the mechanical frequency of the orbit:<sup>2</sup>

$$E_n = -\frac{1}{2}nh\omega_n.$$

- Radiation is emitted or absorbed when the electron jumps discontinuously from one orbit to another, with the frequency given by the orbital energy difference and the Planck-Einstein relation (5).

$$\Delta E = E_{n_2} - E_{n_1} = h\nu.$$

From these hypotheses and modeling the hydrogen atom as an electron in orbit around a nucleus, he derived the Rydberg equation with the Rydberg constant expressed in terms of known quantities from EM. It worked because of quantization given by the Planck-Einstein relation.

But there was more. From these findings Bohr realized that the orbital angular momentum of the electron in the  $n$ th orbit must be an integral multiple of the Planck constant divided by  $2\pi$ , now recognized as the fundamental quantum of angular momentum. And it got a name,  $\hbar$ :

$$\hbar = \frac{h}{2\pi}.$$

These days, it's  $\hbar$  which is used mostly by physicists.

Of course, the meaning of quantum numbers was still unanswered.

High point: The particle nature of radiation is intimately concerned with the explanation of the spectra of heated elements due to atomic structure in terms of nuclei and electrons and the quantization of angular momentum.

### 4. De Broglie and his particle waves

So far, we have considered the use of the Planck-Einstein relation in a classical context. Moving on to Special Relativity, we know that for a massless particle like a photon, the SR energy-momentum dispersion relation

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<sup>2</sup> Looks like the energy of a harmonic oscillator, doesn't it?

$$E^2 = p^2 c^2 + m^2 c^4$$

reduces by using the Planck-Einstein relation to the square of

$$E = pc = h\nu = \frac{hc}{\lambda}.$$

Therefore

$$\lambda = \frac{h}{p}, \quad (6)$$

which is the de Broglie relation. But de Broglie's idea did not stop with massless particles. He used the Lorentz transformation and the Planck-Einstein relation to show that the particle wavelength was given by the same equation for massive particles. This means that they possess wave-like properties – what has become the notion of *wave-particle duality*. Something new has been introduced, again by using  $E = h\nu$ .

De Broglie published his results in 1923 and included them in his PhD thesis in 1924! Years later, he interpreted the wave-particle duality as a particle accompanied by a wave which piloted its trajectory. The idea was developed later into the pilot-wave hypothesis. But that is another kettle of quanta...

High point: Not only does light have both wave and particle properties, so do all particles, even those with mass.

Accumulative high point: In all these instances, the Planck constant plays its role as mediator between energy and frequency.

## 5. Schrödinger and time evolution

Bohr's model of the hydrogen atom gave good results, but its interpretation was fraught. The problem turned out to be that it was based on a combination of classical mechanics and EM plus the Planck-Einstein relation, and these two sorts of beings made for an uncomfortable marriage. The time was ripe for the Austrian physicist Erwin Schrödinger to develop the theory which would subsume the two – wave mechanics. This was in 1926.

Schrödinger based his derivation on the classical wave equation and the classical Hamiltonian -- the equation for the total energy in terms of kinetic and potential energy. But he included the very non-classical Broglie relation and the Planck-Einstein relation for the energy. In spite of its classical antecedents, the resulting equation is pure quantum. Here is Schrödinger's equation:

$$i\hbar \frac{\partial \Psi(x)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi(x) + V(x) \Psi(x), \quad (6)$$

This equation includes an important sub-equation, namely the operator for the momentum:

$$\hat{p} = i\hbar \nabla. \quad (7)$$

Schrödinger then used this equation to analyze and explain the energy spectrum of the hydrogen atom, including explanation of the Bohr equation.

Some explanation is in order. The quantity  $\Psi$  in this equation is called the wave function. (Nowadays, we call that the coordinate representation of the state vector.) The wave function represents the state of the system, the hydrogen atom. The term on the left is the rate of change of the wave function multiplied by  $i$ , the square root of  $-1$ , and  $\hbar$ , which is none other than our old friend the Planck constant divided by  $2\pi$ , which makes the equation prettier. The right-hand side is the total energy as represented by the quantum Hamiltonian, the sum of the kinetic and potential energies

$$H = K + T = \frac{p^2}{2m} + V(x) \quad (8)$$

but with the momentum  $p$  replaced by the momentum operator from (7).

An operator operates on (does something to) a wave function. If the effect is just to multiply it by a number, then that number is a possible value of the physical quantity represented by that operator. For instance,

$$\hat{p}\Psi = i\hbar \nabla \Psi = p_a \Psi \quad (9)$$

means that  $p_a$  is a possible value (called an eigenvalue) of the momentum. Putting the operator (7) into the equation (8) gives the right-hand side of (6). In this case,  $\Psi$  is the eigenfunction of the operator  $\hat{p}$  with eigenvalue  $p_a$ . In addition, the absolute square of the wave function (eigenfunction),  $|\Psi|^2$  is interpreted as the probability that  $\hat{p}$  is of value  $p_a$ . This is the Born rule.

One important thing about the Schrödinger equation is that it gives the rate of change of  $\Psi$  in time (after division by  $i\hbar$ ). In a simpler-looking form

$$i\hbar \frac{\partial \Psi(x)}{\partial t} = H\Psi(x).$$

All this is in terms of operators and wave functions and so is pure QM, no more classical around. And it all started with  $E = h\nu$ .

### Bibliography

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