Note to the reader (assuming there is one...).

This document is just an overview of the subject, composed essentially of notes I have made to remind myself of the math behind the main principles of modern physics. I should say, the mathematical and physical concepts which are the main principles of modern physics. No attempt to teach anything or to indicate any real calculations of cross sections or whatever. It presents, for me, the mathematics of physics, the concepts you need to know in order to understand the principles presented in simpler books, such as for laymen, no more.

Note: The Minkowski metric signature used will be (+1,-1,-1,-1). Among authors cited, only Robinson uses the opposite. See Table 2.
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1. Lie groups

Let’s start with a super-brief overview, to get an idea of where we are going.

1.1. Really short summary

For physicists, a Lie group is a continuous transformation group. Its elements may be transformations such as rotations or translations but not reflections, as these are not continuous. Such groups are abstract things, but we can imagine concrete examples, called representations. Since the Lie group and therefore the representation are continuous, the latter can be generated incrementally from the identity operator by using operators called generators. The generators are derivatives of Lie group elements (in a representation) and form a vector space called a Lie algebra. The generators are especially interesting to physicists because they may represent physical observables such as angular momentum. They thus provide links between transformations and observables. This allows us to deduce properties of the world around us from properties of mathematical groups, which are thus a window onto the world.¹

Now for more details. Take a deep breath...

1.2. Groups and representations

A group is a set of transformations which is closed and invertible. More specifically, a group, call it \((G,\ast)\), is a set of objects, \(G\), and some operation, \(\ast\), on them subject to certain conditions:²

- closure: \(\forall g_1, g_2 \in G, g_1 \ast g_2 \in G\)
- associativity: \(\forall g_1, g_2, g_3 \in G, (g_1 \ast g_2) \ast g_3 = g_1 \ast (g_2 \ast g_3)\)
- identity: \(\exists e \in G, \forall g \in G, e \ast g = g \ast e = g\)
- inverse: \(\forall g \in G, \exists h \in G | g \ast h = h \ast g = e\)

The \(g_i\) are the transformations, the elements of the group. They are in general distinct from the objects transformed. The number of elements is the order, or dimension, of the group. It may be infinite.

There is some inconsistency in the literature. Many authors do not use the term order, which can lead to confusion between the dimension of the vector space of objects acted on and that of the elements of the group. The order, or dimension, of the group is in general not the same as the dimension of the vector space. The group, for instance, \(\text{SO}(2)\) operates on a 2-dimensional vector space but is of group order 1. The group of permutations of three objects, a discrete (non-continuous) group, contains six elements (the permutations) acting on only three objects.

A Lie group is a group which is also a differentiable manifold, meaning that its elements are organized continuously and smoothly (as opposed to elements of discrete groups). For closure, the group operator must induce a differentiable map of the manifold onto itself. Every group element \(A\) induces a map that takes any element \(B\) to another element \(C = A \ast B\), and this map must be differentiable.

Being continuous, a Lie group is parametrized by one or more continuous variables, the number of which is the order of the group.

A representation can be thought of as an instantiation of an abstract group, representing the elements of the Lie group (or algebra) as operators on a vector space. More rigorously, a representation is a map between any

¹ I think Pythagoras would have loved this!
² Math symbols explained in annex 11.

Symmetry, groups and quantum field theory   4   2021-12-13
abstract element \( g \) of a group \( G \) and a linear \textit{transformation} \( R(g) \) of some vector space \( R : g \rightarrow D_R(g) \)
in such a way that the group properties are preserved:

\[
\begin{align*}
D_R(e) &= 1, \\
D_R(g_1)D_R(g_2) &= D_R(g_1 \ast g_2).
\end{align*}
\]

Note that it is the identification of each element of an abstract group manifold with a linear \textit{transformation} of a vector space – an operation in the vector space. The elements of the representation tell how to transform the vector space, for instance, to rotate it or give it a boost. In physics especially, it is convenient to represent the transformations by a set of matrices. A representation is \textit{irreducible} when it is a representation of a group \( G \) on a vector space \( V \) which has no invariant subspace besides the zero space \( \{0\} \) and \( V \) itself. The elements of the representation tell how to transform the vector space, for instance, to rotate it or give it a boost. In physics especially, it is convenient to represent the transformations by a set of matrices. A representation is \textit{irreducible} when it is a representation of a group \( G \) on a vector space \( V \) which has no invariant subspace besides the zero space \( \{0\} \) and \( V \) itself. Different representations of a specific group may (and usually do) have different dimensions.

In order to study groups of transformations on a vector space \( V \), let’s start simply.

\( \mathcal{L}(V) \)
is defined as representing the set of all linear operators on the vector space \( V \). A linear operator is a function \( T \) from \( V \) to \( V \) itself which satisfies the linearity condition:

\[
T(cv + w) = cT(v) + T(w).
\]

Now we will take subsets corresponding to certain conditions and then subsets of those subsets to reach the groups of interest in physics.

The most general group will take any point in \( V \) to any other point in \( V \). If \( V \) has dimension \( n \), then the group elements can be represented by \( n \times n \) matrices and these must be non-singular, or \textit{invertible}. Such a group is the \textit{largest} and most general Lie group in \( n \) dimensions. This group is defined as the \textit{general linear group}\( GL(n) \), denoted by \( GL(n) \).

\[
GL(n) \subset \mathcal{L}(V).
\]

Then, according to whether the elements are real or complex, we have the \textit{real} or \textit{complex general linear group in \( n \) dimensions}.

\[
GL(n) \rightarrow GL(n, \mathbb{R}) \quad \text{(real, \( n \)-dimensional field)}
\]
\[
GL(n) \rightarrow GL(n, \mathbb{C}) \quad \text{(complex, \( n \)-dimensional field)}
\]

If the determinant of each element is \( +1 \), these groups become the \textit{special linear groups}

\[
SL(n, \mathbb{R}) \quad \text{or} \quad SL(n, \mathbb{C}).
\]

Going further, consider subgroups of these operating on vector spaces possessing a \textit{non-degenerate Hermitian form} \((\cdot, \cdot)\), a function which assigns a scalar value to an ordered pair of vectors. Examples are the inner or scalar product, and the Minkowski metric. The set of isometries \( Isom(V) \), consists of operators \( T \) which “preserve” \((\cdot, \cdot)\), meaning that

\[
(Tv, Tw) = (v, w) \quad \forall \ u, w \in V. \tag{1.1}
\]

If the condition

---

3 We are leaving out an awful lot here, such as the definition of an invariant subspace.
4 Jeevanjee, 25.
5 Jeevanjee, 118; Robinson, 76.
6 Jeevanjee, 118. The scalar field \( \mathbb{C} \) has only its transformation properties which are concerned by the group.
7 Or a one-form operating on a vector …?
for all $v \in V$, with $v \neq 0$ \hspace{1cm} (1.2) \\
holds (it’s positive-definite), then $\langle \cdot, \cdot \rangle$ is called the inner product and the vector space is an inner-product space. If the inner product is interpreted as a length, then an isometry can be seen as preserving lengths. It is also a group. Now things are looking interesting.

From here on, let’s consider only isometries and apply them to three different vector spaces. These are all subsets of $n$-dimensional general linear groups. Since the conditions on each reduce the number of degrees of freedom and so the number of independent variables, these groups will not have the same order, or group dimension, as the dimensions of the vector space where they operate.

1) If $V$ is a real inner-product space, then in an orthonormal basis, an isometric transformation operator $T$ must obey

\begin{equation}
T^T T = 1 \quad \text{or} \quad T^T = T^{-1},
\end{equation}

which is the orthogonality condition. (Since the $T$ are matrices, $1$ must be the identity matrix.) This is easily seen, as a transformation of an inner product takes place as follows.

$$v'^2 = v'^T v' = v^T O v = v^2.$$ 

2) If instead $V$ is a complex inner-product space, we find

\begin{equation}
T^\dagger T = 1 \quad \text{or} \quad T^\dagger = T^{-1},
\end{equation}

where $T^\dagger$, the adjoint (or Hermitian conjugate), is the transpose of the complex conjugate of the matrix. Such operators are termed unitary and in an orthonormal basis are represented by unitary matrices. Obviously, a unitary operator is an isometry of both a real and a complex inner-product space.

3) The vector space $V$ may be a real vector space with a Minkowski metric $\eta$, 

$$\eta(v_1, v_2) = \eta(v_0 v_0 - v_1 v_1 - v_2 v_2 - v_3 v_3...),$$

which is a non-degenerate Hermitian form, but not an inner product, since it is not positive-definite. It could have any number of negative and positive terms. Of course, we are interested in the 4-dimensional spacetime case (1.4). Then its group of isometries must satisfy

\begin{equation}
\Lambda^T \eta \Lambda = \eta,
\end{equation}

where $\Lambda$ is the 4-dimensional Lorentz transformation. In fact, this can be taken as a definition of the Lorentz transformations.

- The Lorentz group is then the set of all transformations which preserve the metric of Minkowski space. It is denoted by $O(1,n-1)$ in this convention, so $O(1,3)$ is the 4-dimensional Minkowski spacetime of Special Relativity (SR).

Summary: The matrix representations of isometric (distance-preserving) subgroups of the general linear group $GL(V)$, acting on the $n$-dimensional vector space $V$, are the orthogonal or unitary matrices, and the Lorentz transformations $- O(n)$, $U(n)$ and $O(1, 3)$. In general, the parameter $n$ is not the order, or dimension, of the group. We will see that $SO(3)$ has a dimension of 3, but $SO(2)$ of only one.

Both $O(n)$ and $U(n)$ have subgroups characterized as “special”, meaning that they contain only those matrices whose determinant is +1.

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8 Jeevanjee, 34.
9 I think...
10 Jeevanjee uses the unusual (to me) convention of $O(n-1,1)$, putting time at the end, for some reason beyond my fathoming.
They are called the \textit{special orthogonal group}, \(SO(n)\), and the \textit{special unitary group}, \(SU(n)\).

2. Generators and Lie algebras
The generators of a Lie group make up the Lie algebra of the group.

2.1. Lie algebras
The transformation corresponding to a Lie group is specified by a number of parameters. For instance, rotation in 2-dimensional real space, \(SO(2)\), is specified by a single parameter, the angle of rotation, and so is of order 1. So the number of parameters is not necessarily the dimension of the space acted upon by the group. Since it is continuous, a \textit{representation} of a Lie group can be generated by infinitesimal operations starting from the identity element. So taking each parameter \(\delta\theta_a\) as an infinitesimal increment of parameter \(\theta_a\)

\[
D(\delta\theta_a) = I + i\delta\theta_a T^a,
\]

where the factor \(i\) is included so the \textit{generator} \(T^a\) in the representation will be Hermitian. Note that as soon as we talk about generators, we are referring to a representation of a Lie group, not the abstract group. Applying a generator multiple (infinite) times leads to Lie group elements in the representation:

\[
D(e^{i\theta_a T^a}) = e^{i\theta_a T^a}.
\]

Here, \(\theta_a\) is a parameter, and \(T^a\) is a generator of the Lie group and a member of the Lie algebra of the Lie group (definition coming). There is one generator per parameter, so \textit{the number of generators is constant for a given group}, even though the dimension of a representation and so that of \textit{each matrix} \(T^a\) in the representation may vary. You may soon tire of hearing this repeated, but it is important to distinguish between the order, the dimension, of the group and the dimension of the representation, which may vary. Examples of representations of different dimensions of the group \(SU(2)\) are shown in Table 1.

From equation (2.2), we see that the generator, \(T^a\) is basically the derivative evaluated at \(\theta_a = 0\):

\[
T^a = -i \frac{\partial D}{\partial \theta_a} \bigg|_{\theta_a=0}.
\]

Alternatively, from (2.1) and (2.3), one can “see” that a Lie algebra is the \textit{tangent space} to the group at the identity.\footnote{Meaning the space of tangents to elements of the group.}

As an example, consider the case of \(SO(2)\), rotations in 2 dimensions, with only a single parameter, the angle:

\[
D(\theta) = \begin{cases} \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \end{cases} \bigg| \theta \in [0, 2\pi).
\]

Then

\[
-i \frac{dD}{d\theta} \bigg|_{\theta=0} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = T_R,
\]

is the generator, a 2x2 matrix. For the starting vector \(r_0 = (1, 0)\),

\[
Xr_0 = i \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\]
which is a vector pointing in the direction of the change in \( r_0 \) under a rotation. It is evident that this works also for \( r \) at \((0,1), (-1,0)\) and only slightly less evident for arbitrary angles.

Formally, a **Lie algebra**\(^{12}\) is a vector space \( g \) together with a binary operator, called the **Lie bracket**, 
\[ [\cdot, \cdot] : g \times g \rightarrow g \]  

The binary operator satisfies the following conditions:

- **Bilinearity:** \([aX + bY, Z] = a[X,Z] + b[Y,Z]\) and \([Z, aX + bY] = a[Z,X] + b[Z,Y]\) , for arbitrary numbers \(a, b\) and for \(X,Y,Z \in g\).
- **Anticommutativity:** \([X,Y] = -[Y,X] \forall X,Y \in g\).
- The Jacobi identity: \([X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] = 0 \forall X,Y,Z \in g\).

The Lie bracket is not necessarily associative. The **Lie bracket** \([\cdot, \cdot]\) tells us how to combine these matrices. A Lie algebra is “closed under commutators”\(^{13}\): If \(X\) and \(Y\) are elements of a Lie algebra, then so is 
\[ [X,Y] = XY – YX. \]

Starting with the Jacobi identity, one can show that the commutator of the generators, \(X^i\), obeys
\[
[X^i, X^j] = i f^{ij}_k X^k,  \tag{2.5}
\]
where the **structure constants** \( f^{ij}_k \) of the Lie algebra are independent of the representation, even though the elements \(X^i\) are not.\(^{14}\) This may seem confusing until you recognize that one of the generators \(X^i\) is really a matrix with elements \([X^i]_{\alpha\beta}\) where subscripts \(\alpha\) and \(\beta\) for the matrix elements run from 1 to \(n_R\), the dimension of the representation, which is arbitrary; whereas the generator superscript \(i\) runs from 1 to the order (dimension) of the group, which is constant and a property of the group. There is therefore a fixed number of structure constants for each group. For instance, we will see that the structure constants for \(SO(3)\) and \(SU(2)\) are the Levi-Civita symbols.

- The **Lie algebra** is composed of the generators, which are subject to the Lie bracket defined by the structure constants. “The Lie algebra is encoded within the commutation relations of the generators.”\(^{15}\)

The elements of the Lie algebra are operators (matrices), not numbers, and they do not commute. So although closure requires
\[
e^{i\alpha_i X_i} e^{i\beta_j X_j} = e^{i\delta_k X_k},
\]

we cannot do a simple sum of the exponents of the left side of the equation. Instead, we can use power series to derive the **Baker-Campbell-Hausdorff formula**:
\[
e^{i\alpha_i X_i} e^{i\beta_j X_j} = e^{i(\alpha_i X_i + \beta_j X_j) - \frac{1}{2} [\alpha_i X_i, \beta_j X_j],} \tag{2.6}
\]
If we think of the *parameter space* of the group as a *vector space*, in which each point designates a specific element of the group, then the generators specify the space near the identity. Like the 3-d unit vectors \(\hat{i}, \hat{j}\) and \(\hat{k}\), which are also defined near the identity (zero) but form a set of basis vectors, the generators form a basis for the entire vector space of the parameters. Said the other way around, the parameters specify a point in the vector space in terms of the generators. Such a point in the vector space corresponds to a specific element of the group, such as a rotation through a certain angle.

Once again, the number of generators is equal to the number of parameters and is always the same for a given group, regardless of the dimension of the representation.

\(^{12}\) Schwichtenberg, PS, 45.
\(^{13}\) Schwichtenberg, PS, 155.
\(^{14}\) Maggiore, 15.
\(^{15}\) Blundell and Lancaster, 84.
A **Casimir element** $C$ is built from elements of the Lie algebra in such a way that it commutes with every
generator $X$ of the group.

$$[C, X] = 0. \tag{2.7}$$

Schur's lemma then says that it must be a multiple of the identity, so Casimir elements provide linear operators
with a constant value for each representation. They therefore can be used to label the representation. An example is $J^2 = \sum_i J_i^2$ for rotations.

Within a Lie algebra, the set of mutually commuting generators, which therefore can be diagonalized simultaneously, is called the **Cartan subalgebra**. The number of such generators is the **rank** of the subalgebra.

Although a representation may have an arbitrary dimension, the rank is constant across all representations (and Lie algebras) of a Lie group, as is the order of the group. Since the Cartan generators are mutually commuting, they share a set of eigenvectors in the vector space.

The eigenvectors of the Cartan generators span the space and form a basis for the representation. Their eigenvalues can be used to label the corresponding states within the representation, just as the Casimir element provides a label for each representation. Examples will follow.

Among those groups important to physics (more later), $SO(n)$ and $SU(n)$ apply to spaces of $n$ real or complex dimensions. This $n$ is not the order of the group, but it does fix (but does not equal) the number of generators. A subset of these are Cartan generators. An arbitrary $SU(n)$ group will always have $n^2 - 1$ generators and be of rank $n - 1$, whereas an arbitrary $SO(n)$ group will have $\frac{n(n-1)}{2}$ generators. In different representations, the number of generators remains the same although the dimension of the representation may vary.

For example: Any representation of $SU(2)$ has 3 generators of which one is the Cartan generator, taken to be $J_3$. For a $j$ representation, where $j$ corresponds to the Casimir operator $J^2$ of value $j(j + 1)$ (not to the dimension of the representation, which is $2j + 1$), the generators are $(2j + 1) \times (2j + 1)$ matrices. The eigenvalues of the Cartan generator run in integral increments from $-j$ to $+j$. Although the number of Cartan operators is constant across representations, the number of their eigenvectors depends on the dimension of the representation.

A special representation, the **adjoint representation**, may be defined in term of the structure constants by

$$[T^a]_{bc} = -if_{abc}. \tag{2.8}$$

This definition has two important results:

- The adjoint representation as well as its matrices has the same dimension as the structure constants and, therefore, the dimension (number of parameters) of the group, e.g., $8 \left( n^2 - 1 \right)$ for $SU(3)$ or 3 for $SU(2)$.
  - The adjoint representation is the (only) one in which the number of eigenvectors of the Cartan generators (the simultaneously diagonalizable generators) is the dimension of the group. This is equivalent to saying the number of eigenvectors is the same as the number of generators, which is constant and independent of the representation. The dimension of each eigenvector is that of the representation.
  - So in the adjoint representation, the number of generators = the dimension of the representation = the number of eigenvectors of the Cartan generators = the number of weight vectors (to be defined soon).
- The Jacobi identity then may be used to derive (summed over $c$)

$$[T^a, T^b] = i f_{abc} T^c. \tag{2.9}$$
The fundamental representation of SO(n) or SU(n) is the one consisting of \( n \times n \) matrices.

Now get this. Classical mechanics can be formulated in terms of Poisson brackets

\[
\{f, g\} = \sum_i \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i} \frac{\partial f}{\partial p_i}, \quad f, g \in C(P)
\]

where \( f \) and \( g \) are observables in a 2n-dimensional phase space \( P \) of \( n \) generalized coordinates \( q_i \) and the \( n \) conjugate momenta \( p_i \). The vector space \( C(P) \) is the set of all complex-valued, infinitely differentiable functions on \( P \). The Poisson bracket is obviously antisymmetric and brute force will show that it obeys the Jacobi identity, which means the set of observables constitutes “...one huge Lie algebra.”\(^{16}\)

Taking the Poisson bracket as the Lie bracket of the Lie algebra, it can be the basis for forming a representation of the Lie algebra in a Hilbert space \( \mathcal{H} \). “Thus the set of all observables in quantum mechanics forms a Lie algebra, which is one of our main reasons for studying Lie algebras...”\(^{17}\) You can’t ask for more motivation than that!

2.2. Reminder – The basics of Lie groups

Let’s sum up:

- A Lie group corresponds to a set of continuous transformations which are characterized by a certain number of parameters, e.g., rotation angles, which is always the same for the group and its representations. This is the order of the group. The dimension of an SU(n) group is \( (n^2 - 1) \), 8 for SU(3) or 3 for SU(2); of an SO(n) group, \( \frac{n(n-1)}{2} \).

- From the equation for a generator (2.3), there is one generator per parameter so the number of generators is also constant and is equal to the order of the group. However, ...

- ...different representations of the same group may have different dimensions.

- Repeat: Although a representation may have arbitrary dimensions, it always has the same number of generators.
  - Caveat: Do NOT confuse the order, or dimension, of a group (constant) and the dimension of a representation (variable).

- The subset of the generators which are commuting and mutually diagonalizable are the Cartan generators. Their number, called the rank of the group, is also a constant for all representations.

- The eigenvectors of the Cartan generators span the space of a representation and therefore are of the same dimension as the representation. The eigenvalues can label the corresponding states.

- The adjoint representation is special because it has the same dimension (order) as the group. It can be defined in terms of the structure constants, as in (2.9).

- The fundamental representation of a group SU(n) is the one consisting of \( n \times n \) matrices.

3. Covering groups and subalgebras

A topological space is simply connected if it is path-connected (not disjoint) and every path between any two points can be continuously transformed into any other path between those points. This means that the space can contain no holes. There is an important fact, which we just have to accept:

There is only one simply-connected Lie group corresponding to each Lie algebra.
This “mother” group to potentially many other Lie groups sharing the same Lie algebra is called the covering group and is said to cover the other groups.

Put the other way around, a covering group is the unique simply-connected Lie group corresponding to a given Lie algebra. Any other Lie group which might correspond to this Lie algebra is not simply connected; it is said to be covered by the covering group – group to group.

4. Lie groups for physics

Our general method of study will be

1) start with an example of a group, e.g., 2x2 matrices and rotation;
2) derive the Lie algebra;
3) use the Casimir element and the Cartan subalgebra of the Lie algebra to look at different representations and search for a simply-connected group, the covering group.

In ordinary n-dimensional space, we are interested in transformations which conserve distances between two points – isometries. So orthogonal (real) and unitary (complex) transformations are important. The most interesting Lie groups for physics are \( SO(n) \) and \( SU(n) \). In particular, \( U(1) \), \( SU(2) \) and \( SU(3) \) apply respectively to the EM, weak and the strong forces.

In addition, SR requires the invariance of the interval (the Minkowski metric)

\[
s^2 = t^2 - x^2 - y^2 - z^2,
\]

which is a kind of distance between two points. So we add the O(1,3) group to the list of potentially important groups for physics. It will turn out to be somewhat more complex than this.

4.1. Orthogonal groups

Since an orthogonal transformation is a linear transformation on a real vector space \( V \) that preserves inner products, it is therefore the group of isometries on \( V \) and is defined, as we have seen in (1.3), by the condition,

\[
O^T O = I
\]

and a special orthogonal group by (4.1) and

\[
det(O) = +1.
\]

The first condition guarantees the conservation of lengths

\[
x_1^2 + x_2^2 + x_3^2 + ...
\]

and the second keeps only rotations (not reflections), a rotation being a continuous linear operator which takes orthonormal bases to orthonormal bases.\(^{18}\)

Equation (4.1) for the group element is consistent with (2.2) for the generator as the former means

\[
e^{i\theta X^T} e^{i\theta X} = 1 \Rightarrow X^T = -X.
\]

So we see that the Lie algebra of O(n) is the set of \( n \times n \) antisymmetric matrices.

\(^{18}\) Jeevanjee, 125, 151.
4.2. Unitary groups

A **unitary** transformation is similar to an orthogonal transformation in that it preserves inner products, but in a complex vector space. So, as an isometry, a unitary group’s Hermitian conjugate $U^\dagger$ must satisfy

$$U^\dagger U = 1.$$  \hspace{1cm} (4.3)

A **special unitary group**, also satisfies

$$\det(U) = +1,$$  \hspace{1cm} (4.4)

Every isometry of a complex inner product space is unitary, and vice versa. In an orthonormal basis, a unitary operator is represented by a unitary matrix. It is clear by (4.1) and (4.3) that a unitary operator is an isometry of both real and complex inner product spaces.

For physics, we define the generator as

$$g = e^{i\theta U},$$

so that the fundamental equation for isometries

$$(e^{i\theta U})^\dagger e^{i\theta U} = 1$$

yields

$$U^\dagger = U$$

and the matrices are *Hermitian* and so can represent physical observables.

4.3. 2-dimensional rotations – SO(2) and U(1)

The unitary group $U(1)$ is just multiplication by a phase $e^{i\theta}$ and represents rotation by an angle $\theta$ in the complex plane.

Rotations in two dimensions are the unit circle, $S^1$, and can be represented by 2-d matrices which are elements of either $SO(2)$ or $U(1)$. From (4.1) and (4.2), the SO(2) group can be represented by rotations in terms of the sine and cosine of the angle of rotation.

$$R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}. \hspace{1cm} (4.5)$$

However, constant distances in 2-d can also be represented by unit complex numbers in 1-d. This is the $U(1)$ group. Such a number is represented by

$$R_\theta = e^{i\theta} = \cos(\theta) + i\sin(\theta). \hspace{1cm} (4.6)$$

One can map (4.6) to a real matrix by using the 2-d identity matrix

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \hspace{1cm} (4.7)$$

to show that

$$R_\theta = \cos(\theta) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sin(\theta) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix},$$

which is identical to the equation for 2-d $SO(2)$, showing that there is an **isomorphism** between the two groups.

---

19 Jeevanjee, 120-121.
U(1) and SO(2), denoted
\[ U(1) \cong SU(2). \]
There is a one-to-one, invertible correspondence between their elements. One can “see” them as the same group with different labels for the elements.

### 4.4. 3-dimensional rotations – SO(3) and SU(2)

The conditions (4.1) and (4.2) for an orthogonal group can be satisfied by 3x3 matrices which are simple extensions of equation (4.5), forming a representation of SO(3). Again there is an isomorphism – with SU(2).

#### 4.4.1. SU(2)

Just as SO(2) rotations could be represented in terms of complex numbers by U(1), we would like to describe SO(3) in terms of a unitary group. In this case, we must use 4-d complex numbers called quaternions, as there are no 3-d complex numbers.\(^{20}\) The unit-length constraint reduces the 4 degrees of freedom to three, as needed for 3-d rotations. We can extend the idea of the complex number from one to four dimensions by defining

\[ i^2 = j^2 = k^2 = -1 \]

and

\[ ijk = -1. \]

Quaternions can be written as 2x2 matrices in terms of the basis vectors (not the Pauli matrices)

\[ 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad j = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \text{and} \quad k = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \quad (4.8) \]

Then a unit quaternion may be written as

\[ q = a1 + bi + cj + dk \]

with

\[ q^\dagger q = 1 \]

and

\[ \det(q) = a^2 + b^2 + c^2 + d^2 = 1. \quad (4.10) \]

In this way, the set of unit quaternions can be written as 2x2 matrices which satisfy

\[ \begin{align*}
U^\dagger U &= 1 \\
\det(U) &= 1
\end{align*} \quad (4.11) \]

which fulfills the requirements (4.3) and (4.4) for a unitary group.

If we write a 2x2 matrix as follows, where the parameters are complex numbers,

\[ U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \]

then conditions (4.11) can be used to solve for c and d to give

\[ U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}, \]

\(^{20}\) Schwichtenberg, PS, 2018, 33. Why not?
which contains two complex numbers and so four real ones. But the requirement that the determinant be +1
takes away one degree of freedom, so we are left with three free parameters for the SU(2) group. This is
consistent with the fact that an arbitrary SU(n) group will always have \( n^2 - 1 \) generators and be of rank \( n - 1 \),
whereas an arbitrary SO(n) group will have \( \frac{n(n-1)}{2} \) generators.

In order to associate this with the rotation of a 3-d vector in space, we set \( a=0 \) and identify only the imaginary
part with the 3-d vector
\[
v = x\hat{i} + y\hat{j} + z\hat{k}.
\]
(4.12)

Keeping the imaginary and real parts separate under arbitrary rotations requires transformations of the type
\[
v' = qvq^{-1}.
\]
(4.13)

With this requirement satisfied, unit quaternions can indeed describe 3-d rotations by equation (4.12). And from
(4.8), this gives us a 2-d unitary representation of rotations in three dimensions.

Let’s look at its Lie algebra. Starting with its expression in terms of generators \( J_i \)
\[
U = e^{i\theta J_i},
\]
and applying requirements (4.11), now requires that
\[
J_i^\dagger = J_i,
\]
meaning \( J_i \) is Hermitian (the reason for the factor of \( i \) in the exponent). The second of equations (4.11) requires
a zero trace.\(^{21}\) So the generators of SU(2) must be traceless Hermitian matrices. A possible basis in terms of 2x2
matrices is the triplet
\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]
(4.14)

which are the Pauli matrices. These satisfy the commutation relation
\[
[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k,
\]
(4.15)

so if we define the generators as \( J_i \equiv \frac{1}{2}\sigma_i \), we have
\[
[J_i, J_j] = i\epsilon_{ijk}J_k
\]
(4.16)

which expresses the Lie algebra \( su(2) \) of SU(2) and shows the structure constants to be the elements of the
Levi-Civita symbol.

4.4.2. SO(3)

Looking for the Lie algebra of SO(3), we know that every element \( O \) of the group can be written in terms of a
generator (member of the associated Lie algebra) as
\[
O = e^{i\theta J}.
\]
(4.17)

Putting this together with conditions (4.1) and (4.2) gives
\[
J^T + J = 0 \quad \text{and} \quad Tr(J) = 0,
\]
(4.18)

which can be satisfied by the following generators:\(^{22}\)

\(^{21}\) A corollary to Jacobi’s identity, \( det(e^A) = e^{tr(A)} \), shows that the trace of a determinant 1 matrix must be zero.
\(^{22}\) Schwichtenberg, 46.

Schwichtenberg, PS, 44.
These are the generators of the group SO(3) and so elements of \( \mathfrak{so}(3) \), its Lie algebra, the vector space of 3x3 antisymmetric matrices. They are quite obviously not basis vectors. They can be written compactly using the Levi-Civita symbol:

\[
(J_i)_{jk} = -i \epsilon_{ijk},
\]

which satisfies the Lie brackets

\[
[J_i, J_j] = i \epsilon_{ijk} J_k.
\] (4.20)

Comparison with (4.16) shows the structure constants to be the same as for SU(2). So SO(3) and SU(2) have the same Lie algebra.

A set of basis vectors for the SU(2) generators acting on \( \mathbb{C}^3 \) is:

\[
J_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad J_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\] (4.21)

Now we would like to know which, if either, of SU(2) and SO(3) is the covering group of the other.

### 4.4.3. SU(2) and SO(3)

Starting with a vector (4.12) expressed in terms of the basis matrices (4.8) and applying a rotation around the z-axis

\[
R_z(\theta) = \cos(\theta) \tilde{\mathbf{i}} + \sin(\theta) \tilde{\mathbf{k}},
\]

using (4.12), leads to the result that a rotation of a unit quaternion by an angle \( \theta \) corresponds to a rotation of the corresponding vector by an angle \( 2\theta \).

So for a single complete rotation of the unit quaternion, the vector goes around twice, i.e., a unit vector in SO(3) corresponds to two different unit vectors in SU(2). Stated the other way around, to a given rotation angle of the vector there correspond two different rotation angles of the unit quaternion. For example, a rotation of the unit quaternion by either \( \pi/2 \) or \( 3\pi/2 \) has the same effect as a rotation of the vector by \( \pi \).

More precisely: For the map

\[
\rho : SU(2) \rightarrow SO(3),
\]

which is a homomorphism, "... for every \( R \in SO(3) \) there correspond exactly two matrices in SU(2) which map to R under \( \rho \)."

By (4.10), SU(2) is the three-sphere, \( S^3 \), a 3-dimensional “spherical” space embedded in four dimensions. SU(2) is an isomorphism of \( S^3 \).

Since the three-sphere is easily seen to be simply connected, SU(2) must be the unique covering group for SO(3). SU(2) is said to be the double-cover of SO(3), which is seen as half of SU(2). In a sense, SU(2) is more complete than SO(3).

We will consider another, perhaps more intuitive, geometric way of seeing the correspondence SO(3) to SU(2) in section 4.6.5.

We have now used methodology steps 1) through 3), deriving the Lie algebra from an example (twice, in fact) of

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23 Schwichtenbert, PS, 36-37.
24 Jeevanjee, 176.
25 Jeevanjee, 140.

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a group and then using that to identify the covering group. Now we can use the Lie algebra to consider other representations.

Let’s just note that all these unit-length conserving groups are in fact the same as \( S^m \), the n-spheres:

- \( S^1 \) corresponds to U(1) and SO(2),
- \( S^3 \) corresponds to SU(2) and so to each half of SO(3), SU(2) being the fundamental or covering group.

It would be handy to refer to them as such, \( S^m \), but history has decided otherwise.

### 4.5. Irreducible representations of SU(2).

The irreducible representations are the ones of particular interest. We have seen that since SU(2) is equivalent (isomorphic) to \( S^3 \), the three-sphere, it is the simply-connected group corresponding to this Lie algebra and so is its covering group.

We can build one Casimir element for this Lie algebra, i.e., which commutes with every generator in the group, \( J_i \), and that, as is well known, is

\[
J^2 = J_1^2 + J_2^2 + J_3^2
\]

in three dimensions. From the definitions of the \( J_i \) in (4.21), it is easy to show that

\[
J_{3-d}^2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 2I,
\]

twice the identity matrix. In two dimensions, from the \( J_i \equiv \frac{1}{2} \sigma_i \) and the Pauli matrices (4.14),

\[
J_{2-d}^2 = \begin{pmatrix} 3/4 & 0 & 0 \\ 0 & 3/4 \end{pmatrix} = \frac{3}{4}I.
\]

We see that the representations can be labeled by the Casimir operator values, 2 or \( \frac{3}{4} \).

Similarly, there is one Cartan element, a diagonal operator, which we usually take as \( J_3 \), labeling the element by its eigenvalue \( m \).

Following standard QM methods, we can define two operators

\[
J_+ = J_1 + iJ_2 \quad \text{and} \quad J_- = J_1 - iJ_2
\]

and use the commutation relations (4.16) to show that they are the usual ladder or step (raising and lowering) operators used, e.g., for angular momentum. So we finish with the representations for SU(2) shown in Table 1. The representations are labeled by \( j \), the number associated with the Casimir operator, and the different elements by \( m \), the eigenvalues of the Cartan operator, \( J_3 \).

Looking ahead a wee bit, we can say that particles of different spin belong to different representations of a rotation group.

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26 “Same as” is too vague. Are they in fact homomorphisms or isomorphisms?
### Table 1. Representations of SU(2).

<table>
<thead>
<tr>
<th>Dimension</th>
<th>( j ) (( J^2 ))</th>
<th>Cartan eigenvalues, ( m (J_3) )</th>
<th>( n )-sphere</th>
<th>Casimir eigenvalues</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td></td>
<td>( j(j+1) = 0 )</td>
</tr>
<tr>
<td>2</td>
<td>1/2</td>
<td>-1/2 , 1.2</td>
<td>( S^1 )</td>
<td>( j(j+1) = 3/4 )</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>-1,0,1</td>
<td></td>
<td>( j(j+1) = 2 )</td>
</tr>
<tr>
<td>4</td>
<td>3/2</td>
<td>-3/2, -1/2, 1/2, 3/2</td>
<td>( S^3 )</td>
<td>( j(j+1) = 15/4 )</td>
</tr>
</tbody>
</table>

### 4.6. Lorentz transformations

As already noted (1.5), a Lorentz transformation \( \Lambda \) must conserve the Minkowski metric \( \eta \) and so must satisfy
\[
\Lambda^T \eta \Lambda = \eta. \tag{4.22}
\]

From this alone one can deduce that the determinant of \( \Lambda \) must be \( \pm 1 \) and also that
\[
\Lambda_0^0 = \pm \sqrt{1 + \sum_i (\Lambda_0^i)^2},
\]
meaning \( \Lambda_0^0 \) is either \( \geq +1 \) or \( \leq -1 \). These two constraints together give four combinations of which only one can be generated from the identity element by infinitesimal transformations, as required by a Lie algebra. This is the so-called **proper orthochronous Lorentz group**, also referred to as the **restricted Lorentz group**, represented by the symbol \( \Lambda^\uparrow_+ \), which has determinant \( +1 \) and \( \Lambda_0^0 \geq +1 \). The word “proper” here refers to the \( +1 \) value of the determinant and orthochronous means that the direction of time is not changed. The other three sub-categories can be reached by parity and time-reversal transformations of the restricted Lorentz group, so the entire Lorentz group may be represented by
\[
O(1,3) = \{ \Lambda_+^\uparrow, \Lambda_F \Lambda_+^\uparrow, \Lambda_T \Lambda_+^\uparrow, \Lambda_P \Lambda_T \Lambda_+^\uparrow \}.
\]

Consider an infinitesimal vector transformation:
\[
\Lambda^\uparrow_\nu = \delta^\mu_\nu + \omega^\mu_\nu
\]
and require conservation of a length in Minkowski spacetime
\[
\eta_{\mu\nu} x^\mu x^\nu = \eta_{\mu\nu} (\Lambda^\mu_\rho x^\rho)(\Lambda^\nu_\sigma x^\sigma) = \eta_{\mu\nu} x^\mu x^\nu.
\]

Ignoring square terms in \( \omega^\mu_\nu \) leads to
\[
\omega^\mu_\nu = -\omega^\nu_\mu,
\]
so the Lorentz group is represented by a 4x4 antisymmetric matrix and therefore has six independent parameters. These may be taken as three rotation angles and the three components of the boost velocity \( \vec{v} \).

Rotations are then simply 3-d rotations (4.19) tucked into the Minkowski spatial part:
\[
\Lambda_{rot} = \begin{pmatrix} 1 & 0 \\ 0 & R_{3x3} \end{pmatrix}
\]

---

27 Following treatment based on Schwichtenberg, PS, 66ff.
with generator
\[
J_i = \begin{pmatrix} 1 & 0 \\ 0 & J_i^{3d} \end{pmatrix}, \tag{4.23}
\]
For a boost along the x-axis, the generator defined by
\[
\Lambda^\mu_\rho = \delta^\mu_\rho + \epsilon K^\mu_\rho
\]
must satisfy (4.22), which leads to
\[
K^T \eta = -\eta K. \tag{4.24}
\]
Following the method of (4.23), we write the general generator
\[
X = \begin{pmatrix} X_{00} & a \\ b & X^{3d} \end{pmatrix}
\]
and plug it into (4.22) to show the general form for a generator of O(1,3) to be
\[
X = \begin{pmatrix} 0 & a \\ a & X^{3d} \end{pmatrix}, \text{ with } X^{3d} \in o(3), a \in \mathcal{R}^{n-1}.
\]
Now we can use our previous knowledge of the Lorentz transformations to deduce the form of a generator of a boost along the x-axis, which does not affect the y or z coordinates:
\[
K_1 = \begin{pmatrix} a & b \\ c & d \\ 0 & 0 \end{pmatrix},
\]
Plugging this into (4.24) to find one way to write the first of the following set of generators, the others being found similarly.
\[
K_1 = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_3 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}. \tag{4.25}
\]
A general restricted Lorentz transformation is then of the form
\[
\Lambda = e^{i \vec{J} \cdot \vec{\sigma} + i \vec{K} \cdot \vec{\delta}}. \tag{4.26}
\]
Brute-force calculation shows that the commutation relations of the six generators are
\[
[J_i, J_j] = i \epsilon_{ijk} J_k, \quad [J_i, K_j] = i \epsilon_{ijk} K_k, \quad [K_i, K_j] = -i \epsilon_{ijk} J_k.
\]
Note that the third of these commutators says two boosts are equivalent to a rotation. Combine these generators linearly by defining two complexified generators
\[
N_i^\pm = \frac{1}{2} (J_i \pm i K_i), \tag{4.27}
\]
so that the commutation relations become
\[
[N_i^+, N_j^+] = i \epsilon_{ijk} N_k^+, \quad [N_i^-, N_j^-] = i \epsilon_{ijk} N_k^-, \quad [N_i^+, N_j^-] = 0. \tag{4.28}
\]

28 Schwichtenberg, PS, 68.
Now, $N_+^-$ and $N_-^-$ each satisfy the commutation relations (4.20) for the Lie algebra of SU(2). So the complexified Lie algebra for the restricted Lorentz group consists of two copies of $su(2)$, the Lie algebra for SU(2), which is therefore the covering group for the restricted Lorentz group. We can label the irreducible representations of each of the two SU(2) groups by its Casimir variable $j$, as in Table 1. Denoting the representations by $(\jmath^+, \jmath^-)$, we have the representations $(0,0)$, $(\frac{1}{2},0)$, $(0,\frac{1}{2})$ and $(\frac{1}{2},\frac{1}{2})$, to name only those of particular interest in physics.

We have now found that SU(2) is the covering group both for the special orthogonal group SO(3) and for the restricted Lorentz group $\Lambda^\pm_+$. A very important group indeed!

4.6.1. The Lorentz algebra

From (4.25), the Lorentz boost in 4 dimensional Minkowski space can be written so as to indicate clearly their nature as matrices.

$$[K_1]_{\mu \nu}^\mu = \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ [K_2]_{\mu \nu}^\mu = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ [K_3]_{\mu \nu}^\mu = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}.$$  

Using the metric to lower one index gives, e.g.,

$$[K_1]_{\mu \nu} = \eta_{\mu \rho} [K_1]_{\rho \nu}^\rho = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{ etc.}$$

The $[K_i]_{\mu \nu}$ are thus a set of antisymmetric imaginary matrices. as are the rotation matrices such as

$$[J_1]_{\mu \nu} = \eta_{\mu \rho} [J_1]_{\rho \nu}^\rho = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}.$$  

This also means they are all Hermitian.

We can adopt a different notation for the same thing by assuming the subscript on $J_i$ and $K_i$ as another dimension of the matrices and write $J_{\mu \nu}$ as the generator for rotations between the $\mu$ and $\nu$ dimensions (directions), so that $J^{10}$ is a boost in the x direction and $J^{23}$ a rotation mixing y and z, i.e., around the x-axis. In other words,

$$K^i \rightarrow J^{i0} \quad \text{and} \quad J^i \rightarrow J^{jk}.$$  

Such a general antisymmetric Hermitian matrix can then be written

$$(J_{\mu \nu})^{ab} = -i (\eta^{\mu a} \eta^{\nu b} - \eta^{\nu a} \eta^{\mu b}),$$

as can be derived by brute force. One can then somewhat more laboriously show that

$$[J_{\mu \nu}, J_{\rho \lambda}] = i (\eta^{\lambda \mu} J_{\rho \nu} - \eta^{\nu \mu} J_{\rho \lambda} - \eta^{\rho \mu} J_{\lambda \nu} + \eta^{\rho \nu} J_{\lambda \mu}),$$  

which is therefore the Lie-algebra bracket for the Lorentz group. It defines the Lorentz algebra.

---

29 Complexification due to the factor 1 in equation (4.27).
30 Robinson, 131.
An otherwise arbitrary set of matrices which satisfies the \textit{anticommutation} relation

\[
\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}\mathcal{I},
\]  

(4.30)
is said to constitute a \textbf{Clifford algebra}. If we define

\[
S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu],
\]

(4.31)
with the \(\gamma^\mu\) subject to the Clifford algebra condition, then \(S^{\mu\nu}\) satisfies the Lorentz-algebra bracket (4.29):

\[
[S^{\mu\nu}, S^{\rho\lambda}] = i(\eta^{\lambda\mu} S^{\rho\nu} - \eta^{\lambda\nu} S^{\rho\mu} - \eta^{\rho\mu} S^{\lambda\nu} + \eta^{\rho\nu} S^{\lambda\mu}).
\]

So \(S^{\mu\nu}\) is a general form for generators of the Lorentz algebra – provided it satisfies the Clifford-algebra anticommutation relation (4.31).

### 4.6.2. The (0,0) representation

From Table 1, \(i=0\) for a 1-dimensional representation. This means every matrix is just 1 so nothing changes and this is the scalar representation.

### 4.6.3. The (½,0) representation

Use the Pauli matrices (4.14) as basis matrices, so

\[
N_i^+ = \frac{1}{2}\sigma_i.
\]

From the definitions of \(N_i^+\) and \(N_i^-\), one finds

\[
J_i = \frac{1}{2}\sigma_i, \quad K_i = -\frac{i}{2}\sigma_i
\]

(4.32)
so a general transformation is given by

\[
\Lambda = e^{i\tilde{J}\cdot\tilde{\sigma} + i\tilde{K}\cdot\tilde{\sigma}} = e^{i\tilde{\sigma}\cdot\frac{\tilde{\sigma}}{2} + \tilde{\sigma}\cdot\frac{\tilde{\sigma}}{2}}.
\]

(4.33)
The factors of \(\frac{1}{2}\) show that this is the representation of the double cover of the Lorentz group. Note that these are in terms of complex \(2 \times 2\) matrices. These act on two-component objects called – heads up, here! – \textit{left-chiral spinors}:

\[
\chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}.
\]

(4.34)

Also, note that equation (4.33) is a matrix equation and the operator is defined by its Taylor series

\[
e^M = \sum_{n=0}^{\infty} \frac{M^n}{n!}.
\]

Using the Pauli matrices properties, including

\[
\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \sigma_3^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

the expansion for a boost along the z-axis gives

---

which makes clear the operator is a 2x2 matrix. Similar treatment of a rotation leads to almost the usual matrix in terms of sine and cosine of the rotation angle.

\[
R_3(\theta) = e^{i\theta \frac{\sigma_3}{2}} = 1 + \frac{1}{2} i \theta \sigma_3 + \frac{1}{2} \left( \frac{i}{2} \theta \sigma_3 \right)^2 + \ldots = \begin{pmatrix} \cos \left( \frac{\theta}{2} \right) & -i \sin \left( \frac{\theta}{2} \right) \\ i \sin \left( \frac{\theta}{2} \right) & \cos \left( \frac{\theta}{2} \right) \end{pmatrix}, \tag{4.36}
\]

after using

\[
i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

This shows that spinors are not normal vectors in spacetime because, after a rotation by 2\(\pi\), a spinor changes by a factor of -1. Note that this is due the factor of ½ which comes from the half-integral value of the spin in equation (4.32). It can be detected through experiments studying the interactions of spinors with gauge fields, e.g., in a neutron interferometer.\(^{32}\)

This important point is worth repeating.

A Lorentz transformation rotation by angle \(\theta\) rotates spacetime by angle \(\theta\) but a spinor only by \(\theta/2\). The spinor is rotated not in 4-d spacetime but in 2-d spinor space.

Section 4.6.5 presents a geometric picture of this.

So what do we have? We have oddball thingies which live in a different space from 4-d spacetime because they only rotate half as much as spacetime under rotation. We know this is the \(j=\frac{1}{2}\) representation of SU(2), which behaves like an angular momentum. It is now obvious that this can be considered the \textit{spin} of a spin-½ particle like an electron, whose spin part is in spinor space.

### 4.6.4. The (0,½) representation

By a calculation similar to that of the last section, we can find for this representation that \(J_i\) has the same value, but not \(K_i\):

\[
J_i = \frac{1}{2} \sigma_i, \quad K_i = \frac{i}{2} \sigma_i,
\]

so a general transformation is given by

\[
\Lambda = e^{i \vec{\theta} \cdot \vec{\sigma}} + i \vec{K} \cdot \vec{\phi} = e^{i \vec{\theta} \cdot \vec{\sigma} - \vec{\phi} \cdot \vec{\sigma}/2}. \tag{4.37}
\]

Again, this is a 2 \(\times\) 2 matrix representation which acts on similar but different two-dimensional objects called \textit{right-chiral spinors}:

\[
\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \tag{4.38}
\]

Under rotations, right-chiral spinors transform just like left-chiral spinors by (4.36). But under boosts, they transform by

\[
B_3(\phi) = e^{-\phi \frac{\sigma_3}{2}} = 1 - \frac{1}{2} \phi \sigma_3 + \frac{1}{2} \left( \frac{1}{2} \phi \sigma_3 \right)^2 + \ldots = \begin{pmatrix} e^{-\frac{\phi}{2}} & 0 \\ 0 & e^{\frac{\phi}{2}} \end{pmatrix}.
\]

\(^{32}\) Schwichtenberg, QFT, 91, note 38.
Left and right-chiral spinors transform differently by a negative sign of the exponent under boosts along a given direction. A parity transformation, or mirroring, in the same direction, reverses the sign in the same way. So a right-chiral spinor mirrored along a given direction transforms under a boost along the same direction (and by a rotation) like a left-chiral one, and vice versa.\textsuperscript{33} We will see that this requires both right and left-chiral spinors in a valid description of spin-$\frac{1}{2}$ particles.

The generic name for both types of spinors is \textit{Weyl spinors}.

### 4.6.5. Geometric picture of spin

A kind of stereographic projection, or map, is a heuristic means of relating a space rotation to a spinor.\textsuperscript{34}

Imagine a sphere of radius 1 centered at the origin and describe any point on the sphere by the usual polar-coordinate angles, $\theta$ and $\phi$, the angles with respect to the $z$ axis and to the $x$ axis in the $x$-$y$ plane. Map a point on the sphere to the $x$-$y$ plane through the origin (center of the sphere) by drawing a ray from the top of the sphere, at $x=y=0$ and $z=1$, through the point to be mapped, $P$, and onto point $P'$ in the $x$-$y$ plane. Then consider the plane to be the complex plane with $x$ as its real part and $y$ as the imaginary one, so that the projected point is given by one complex number.

![Figure 1. Bloch sphere construction of spinor space.](image)

Since the triangle $OZP$ is isosceles, the “radius” from the origin to the projected point $P'$ is

$$r = r(\theta) = \tan\left(\frac{1}{2} (\pi - \theta)\right) = \cot\frac{\theta}{2},$$

This maps the point $(\theta,\phi)$ on the unit sphere to

$$z = r(\theta)e^{i\phi} = e^{i\phi}\cot\frac{\theta}{2}.$$

This may be written as

$$z = \frac{\alpha}{\beta},$$

where $\alpha$ can be taken as real and all the complex part is relegated to $\beta$. Then we may pick

$$\alpha = \cos\frac{\theta}{2} \quad \text{and} \quad \beta = e^{-i\phi}\sin\frac{\theta}{2}.$$

Now let’s interpret these numbers as spinor components, which makes the map of the unit sphere to spinor

\textsuperscript{33} Schwichtenberg, NNQFT, 88.
\textsuperscript{34} Robinson, 152ff.
space complete:

\[(\theta, \phi) \rightarrow \psi^*_R = \left( \begin{array}{c} \cos \frac{\theta}{2} \\ e^{-i\phi} \sin \frac{\theta}{2} \end{array} \right),\]

so that

\[\psi_R = \left( \begin{array}{c} \cos \frac{\theta}{2} \\ e^{i\phi} \sin \frac{\theta}{2} \end{array} \right).\] (4.39)

This indeed can be shown to Lorentz transform like a right-chiral spinor.

Spin “up” then means \((\theta, \phi) = (0, 0)\) and this gives the spinor

\[\psi_R = \left( \begin{array}{c} 1 \\ 0 \end{array} \right) = |\uparrow\rangle = |\uparrow\rangle,

whereas “down” means \((\theta, \phi) = (\pi, 0)\), which gives

\[\psi_R = \left( \begin{array}{c} 0 \\ 1 \end{array} \right) = |\downarrow\rangle = |\downarrow\rangle.

The x and y directions can be calculated also, with the appropriate results. The factor of \(\frac{1}{2}\) the angle is made clear from the projection.

*The up and down spin directions are perpendicular (orthogonal) in spinor space but correspond to \(+z\) and \(-z\) in spacetime, which are \(\pi\) radians apart.*

So we find again that a complete rotation in spacetime corresponds to a rotation of only \(\pi\) radians in spinor space. A complete rotation in spinor space corresponds to two in spacetime. This 2-to-1 relation between SO(3) and SU(2) is interpreted to mean that SU(2) is the **non-trivial double cover** of SU(3). The projection also shows that spin is a rotation in spinor space (the complex plane), not ordinary spacetime.

*The stereographic projection provides a geometric picture of spin and the way a direction in space (the sphere), described by SO(3), corresponds to a direction in spinor space (the complex plane), described by SU(2).*

Applying a **parity transformation** to the point on the complex plane,

\[\theta \rightarrow \pi - \theta, \ \phi \rightarrow \phi + \pi,\]

leads to the equivalent left-handed spinor

\[\psi_L = \left( \begin{array}{c} e^{-i\phi} \sin \frac{\theta}{2} \\ -\cos \frac{\theta}{2} \end{array} \right),\]

which, as expected, transforms like a left-handed spinor.\(^3\)

4.6.6. Weyl and Dirac spinors

Remember that in physics, objects like scalars and vectors are defined by their transformation properties. This is also the case for spinors. Spinors are new and rather strange beasts. They seem to live in two spaces at once, with one foot (so to speak) in 4-d Minkowski space (since they change under Minkowski rotations and boosts) and the other in **two-dimensional spinor space** (since they have their own transformation rules). More

---

\(^3\) Robinson, 154-61, milks this Bloch sphere projection for more information, including taking a complex conjugate to go from right to left-handed spinors and back. We will see this another way in the next section.
precisely, they live in a spinor space, one of which is attached to each point of spacetime. Equations (4.35) and (4.36) show clearly that under transformations of Minkowski space, neither left-chiral nor right-chiral spinors transform like 4-vectors.

Let’s change our notation a bit. A left-chiral spinor will be $\psi_L$ and a right-chiral, $\psi_R$. Since spinors live in their own space, spinor space, they have their own spinor metric,

$$\epsilon^{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma^2,$$  \hspace{1cm} (4.40)

which is real. Note that the term on the right is the second Pauli matrix, not a squared term. This is not the Minkowski metric of spacetime but is used similarly, e.g., to raise or lower indices

$$\psi^a = \epsilon^{ab}\psi_b.$$  

This is always possible because $\epsilon$ is invariant under Lorentz transformations in the spinor representation. Thus, a superscripted spinor involves the spinor metric just as a superscripted 4-vector involves the Minkowski metric,

$$x^\mu = \eta^{\mu\nu}x_\nu.$$  

Now that we can raise and lower indices, let’s define super- and sub-scripted spinors as follows:

$$\psi_L := \psi_a \quad \text{and} \quad \psi_R := \psi^a.$$  \hspace{1cm} (4.41)

Note well the dot over the subscript or superscript which means right-chiral; no dot, left-chiral.

Now define two new spinor quantities:

$$\tilde{\psi}_L = \epsilon \psi_L = i\sigma^2\psi_L^* \quad \text{and} \quad \tilde{\psi}_R = -\epsilon \psi_R^* = -i\sigma^2\psi_R^*.$$  \hspace{1cm} (4.42)

Then $\tilde{\psi}_L$ Lorentz transforms like a right-chiral spinor and $\tilde{\psi}_R$ like a left-chiral spinor. Indeed, we may say they are those spinors, for if

$$\psi_L = \tilde{\psi}_R = -i\sigma^2\psi_R^*,$$

then from this and (4.42),

$$\tilde{\psi}_L = i\sigma^2\psi_L^* = i\sigma^2(-i\sigma^2\psi_R^*)^* = i\sigma^2(-i\sigma^2)\psi_R = \psi_R,$$

since $i\sigma^2$ is real and

$$(-i\sigma^2)(i\sigma^2) = I.$$  

The conversion from $\psi_L$ to $\tilde{\psi}_L$ is taken to indicate the operation commonly called charge conjugation (reversal),

$$\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \rightarrow \tilde{\Psi} = \begin{pmatrix} \tilde{\psi}_L \\ \tilde{\psi}_R \end{pmatrix} = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix},$$

although in fact what it does is transform a left-chiral into a right-chiral.

Also,

$$\psi_R = \psi^a = \tilde{\psi}_L = \epsilon(\psi_a)^*$$

36 A spinor space may be viewed as a parametrized family of spaces, $S_x$, each isomorphic to SU(2) and one for each point $x$ in Minkowski space $B$. So, in the lingo of topology, $S_x$ is a fiber “bundled” to Minkowski space $B$.

37 It is customary to use Roman indices for spinors, preferably lower-case (a, b, c...), in order to avoid confusion with $I, j, k$ of basis vectors of Euclidean space.

38 Robinson, 135.

39 I may have the signs backward, but they cancel out in the end.
shows that since the metric raises or lowers the index, complex conjugation converts inverts the chirality; it exchanges dotted and undotted. This is consistent with the notation of (4.41).\textsuperscript{40} So, to take home:

- Complex conjugation interchanges dotted and undotted indices (inverts the chirality).
- The metric raises and lowers indices.
- Transforming a left-chiral spinor into a left-chiral requires both actions.

The spinor metric can be used to construct the inner product

$$\psi_\alpha \psi^\alpha = \psi_\alpha \epsilon^{ab} \psi_b,$$

which is Lorentz invariant (by (4.35) and (4.36)). In addition, terms like

$$\psi^\dagger \psi = \psi_b \epsilon^{ba} \psi_a,$$  \hspace{1cm} (4.43)

are then also invariant, but only because one of the spinors is its complex conjugate.

Let's look at the spinor Lorentz transformations in all their grisly detail:

$$\mathcal{X'} = \mathcal{X}_a', = \Lambda^b_a \mathcal{X}_b = (e^{i\theta \cdot \hat{\sigma}_x + \phi \cdot \hat{\sigma}_y}) \mathcal{X}_b,$$

for left-chiral spinors and

$$\xi' = \xi'^{ia} = \Lambda^{b}_i \xi^b = (e^{i\theta \cdot \hat{\sigma}_x - \phi \cdot \hat{\sigma}_y}) \xi^b,$$

for right-chiral spinors. So finally, the spinor Lorentz transformation operators are as follows:

$$\Lambda^{(\frac{1}{2},0)} = (e^{i\theta \cdot \hat{\sigma}_x + \phi \cdot \hat{\sigma}_y}) := \Lambda^b_a$$  \hspace{1cm} (4.44)

$$\Lambda^{(0,\frac{1}{2})} = (e^{i\theta \cdot \hat{\sigma}_x - \phi \cdot \hat{\sigma}_y}) := \Lambda^b_a$$  \hspace{1cm} (4.45).

Remember that, because of the $\sigma$ terms, these are matrix equations and applying the operators generally requires expressing the exponential function as a power series. And don’t miss the dot on the superscript $a$ on the last term of the second equation.

Under a parity transformation, the generator $J_i$ is unchanged (angular momentum being a pseudovector), but $K_i$ changes sign. This means that, from equation (4.27),

$$N_i^+ \leftrightarrow N_i^-.$$

One more important point: In order to maintain the validity of equations under a parity transformation, we need to have both a left-chiral and a right-chiral spinor. This is rather like in SR, where we are required to to use 4-vectors which describe space and time simultaneously. In the spinor case, the solution adopted is to use a \textbf{Dirac spinor}

$$\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} \psi_a \\ \psi^a \end{pmatrix},$$

where a parity transformation gives

$$\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \rightarrow \Psi' = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}.$$

In spite of appearances, a \textbf{Dirac spinor} is a four-component object, composed of two two-component Weyl spinors.

---

\textsuperscript{40} Dixit Schwichtenber, NNQFT, 99-100.
\[ \Psi = \begin{pmatrix} \psi_L \\
\psi_R \end{pmatrix} = \begin{pmatrix} \psi_{L1} \\
\psi_{L2} \\
\psi_{R1} \\
\psi_{R2} \end{pmatrix}. \]

Its four components exist in spinor space, not 4-d spacetime, and it transforms like two spinors, not like a 4-vector. In math speak, it is said to be in the \((\frac{1}{2}, 0) \oplus (0, \frac{1}{2})\) product representation. In particular, a boost along the z-axis has the formalism

\[
B_3^D(\phi) = \begin{pmatrix} e^{\frac{\phi}{2}} & 0 & 0 & 0 \\
0 & e^{-\frac{\phi}{2}} & 0 & 0 \\
0 & 0 & e^{-\frac{\phi}{2}} & 0 \\
0 & 0 & 0 & e^{\frac{\phi}{2}} \end{pmatrix} = \begin{pmatrix} \Lambda(\frac{1}{2}, 0) & 0 \\
0 & \Lambda(0, \frac{1}{2}) \end{pmatrix},
\]

which does not look at all like the Lorentz transformation of a 4-vector in terms of hyperbolic functions.

### 4.6.7. The \((\frac{1}{2}, \frac{1}{2})\) representation

An object in this representation has two indices, each one transforming under its own 2-dimensional copy of the Lie algebra \(su(2)\), so the object is 4-dimensional. Using the Pauli matrices plus the identity as basis 2x2 matrices, such an object can be written as:

\[
v_{ab} = v_\nu \sigma^\nu_{ab} = v_0 \begin{pmatrix} 1 & 0 \\
0 & 1 \end{pmatrix} + v_1 \begin{pmatrix} 0 & 1 \\
1 & 0 \end{pmatrix} + v_2 \begin{pmatrix} 0 & -i \\
i & 0 \end{pmatrix} + v_3 \begin{pmatrix} 1 & 0 \\
0 & -1 \end{pmatrix}. \tag{4.46}
\]

Going through the separate transformations shows that the resulting transforms could equally well be expressed by a 4-vector formalism, such as this boost along the z-axis:

\[
\begin{pmatrix} v_0' \\
v_1' \\
v_2' \\
v_3' \end{pmatrix} = \begin{pmatrix} \cosh(\phi) & 0 & \sinh(\phi) \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\sinh(\phi) & 0 & \cosh(\phi) \end{pmatrix} \begin{pmatrix} v_0 \\
v_1 \\
v_2 \\
v_3 \end{pmatrix}.
\]

So the \((\frac{1}{2}, \frac{1}{2})\) representation represents vectors and we can use the simpler vector matrix algebra. In fact, a 4-vector is a rank 2 spinor and so has two Cartan generators and two eigenvalues. Whereas a rank-2 tensor has two vector indices, we see now that a rank 2 vector has two spinor indices. So spinors are more fundamental than 4-vectors, since 4-vectors are not appropriate for describing all physical systems on a fundamental level (electrons, for instance).\(^{41}\) Hence the (approximate) expression that a vector is the square root of a rank-2 tensor and a spinor is the square root of a vector. As Robinson says: "... the \((\frac{1}{2}, \frac{1}{2})\) representation ... is the fundamental defining vector representation of the Lorentz group in 1+3 spacetime dimensions."\(^{42}\)

Also, note that the determinant of \((4.46)\) is

\[
det(v_{ab}) = (v_0^2) - (v_1^2) - (v_2^2) - (v_3^2),
\]

the invariant spacetime "distance" of SR.

---

\(^{41}\) Schwichtenberg, 83.

\(^{42}\) Robinson, 130.

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4.7. The Poincaré group

The full spacetime symmetry group of nature is the **Poincaré group**, the Lorentz group plus translations, or rotations plus boosts plus translations. Infinitesimal translations are represented by the generators

\[ P_i = -i \partial_i. \]

The Poincaré group has two Casimir operators, of which the more useful\(^\text{43}\) is

\[ P_\mu P^\mu := m^2 \]

(Yes, that m.). Also, the number \( j_1 + j_2 \) of the representation \((j_1, j_2)\) can be used as a label and is called the spin, so the \((0, 0)\) representation is the spin 0 representation; the \((\frac{1}{2}, 0)\) and \((0, \frac{1}{2})\), spin \( \frac{1}{2} \); the \((\frac{1}{2}, \frac{1}{2})\), spin 1.

5. Finding Lagrangians

Standard notation, at least for us, uses \( \Phi \) for scalar fields, \( \Psi \) for spinor fields and \( A \) or \( \mathcal{V} \) for vector fields.

One of the goals of these studies of groups is to derive proper Lagrangians for the systems of physics. Another is to identify behavior of intrinsic properties of particles like spin, isospin or color, and to infer the existence of mediator particles, or gauge fields. Although no reason is known for it, aside from the fact that the results give the correct equations of motion, two general principles of Lagrangian construction must be observed:\(^\text{44}\)

1. The Lagrangian may only contain the lowest non-trivial derivatives, meaning first or second order. Sometimes the second order is necessary in order to maintain Lorentz invariance.
2. For free fields or particles, we must stop at second order in the field.

The overriding central requirement is, of course, that the action must be Lorentz invariant, which will be satisfied if the Lagrangian is Lorentz invariant (although this is not a necessary condition), for instance, if it is a scalar.

Note that Lagrangians for fields are functions of the components of the field, not of momenta and coordinates of particles. More on this in section 6.3.

We must consider several cases.

5.1. Scalar particles

Scalar fields transform according to the \((0,0)\) representation of the Lorentz group. A Lagrangian (density) obeying the above rules would be of this form:

\[ \mathcal{L} = A \Phi^0 + B \Phi^2 + D \partial_\mu \Phi + E \partial_\mu \Phi \partial^\mu \Phi + F \Phi \partial_\mu \Phi + G \Phi \partial_\mu \partial^\mu \Phi. \]

Consider the various terms:

- The A term is just a constant and so has no effect on the equations of motion.
- Odd powers of \( \partial_\mu \) are forbidden because they are not Lorentz-invariant, so the D term goes out. The F term is odd in terms of both the field and the derivative, so out with it.
- The B term can be ignored as it becomes a constant after use of the Euler-Lagrange equations and so changes nothing physical.
- After integration by parts, assuming the fields go to zero at infinity, a term like \( G \Phi \partial_\mu \partial^\mu \Phi \) would be just like the F term and so is redundant.

\(^{43}\) Schwichtenbert, PS, 89-90.
\(^{44}\) Schwichtenberg, PS, 97-98.
This leaves us with only

\[ \mathcal{L} = C\Phi^2 + E\partial_\mu\Phi\partial^\mu\Phi. \quad (5.1) \]

Another approach to Lorentz invariance could start with the SR dispersion relation

\[ E^2 = p^2 + m^2 \rightarrow -h^2\nabla^2 + m^2. \]

This suggests that we do something like take the square of the Schrödinger equation. If we do that and put in the QM operator for the momentum, the result looks like (5.1), which gives us an identification for the variables C and D. Therefore, we can say the Lagrangian for scalars is

\[ \mathcal{L} = \frac{1}{2}(\partial_\mu\partial^\mu\Phi - m^2\Phi). \quad (5.2) \]

From this Lagrangian, the Euler-Lagrange equations give

\[ (\partial_\mu\partial^\mu + m^2)\Phi = 0, \quad (5.3) \]
the Klein-Gordon equation.

### 5.2. Spin \( \frac{1}{2} \) particles – fermions

By similar but somewhat more laborious reasoning based on Lorentz invariance of Dirac spinors, we find the Lagrangian for the \((\frac{1}{2}, 0) \oplus (0, \frac{1}{2})\) representation to be of the form

\[ \mathcal{L} = A\Psi^\dagger\gamma_0\Psi + B\Psi^\dagger\gamma_\mu\gamma^\mu\partial_\mu\Psi = A\Psi\Psi + B\Psi\gamma^\mu\partial_\mu\Psi. \]

The 4x4 \( \gamma^\mu \) matrices are defined in terms of the identity and the Pauli matrices \( \sigma_i \)

\[ \gamma^0 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \quad \gamma_i = \begin{bmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{bmatrix}. \]

This is the Dirac representation. With \( A = -m \) and \( B = I \), this gives

\[ \mathcal{L} = -m\bar{\Psi}\Psi + i\bar{\Psi}\gamma^\mu\partial_\mu\Psi = \bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi \quad (5.4), \]
the Dirac Lagrangian. A more direct (and believable) “derivation” of the Dirac equation will be shown in section 6.8.

### 5.3. Spin 1 particles

From the \((\frac{1}{2}, \frac{1}{2})\) representation, similar considerations lead to a general invariant form

\[ \mathcal{L}_{\text{Proca}} = C_1\partial^\mu A^\nu\partial_\mu A_\nu + C_2\partial^\mu A^\nu\partial_\nu A_\mu + C_3 A^\mu A_\mu + C_4\partial^\mu A_\mu, \]
where \( A \) is an arbitrary, as yet unidentified vector field. We can ignore the 4\(^{th} \) term, as it does not affect the equations of motion. Passing all this through the Euler-Lagrange equations gives

\[ 2C_3 A^\rho = 2C_1\partial_\sigma\partial^\sigma A^\rho + 2C_2\partial^\rho(\partial_\sigma A^\sigma). \]

Adjusting the constants \( (C_1 = -C_2 = 1/2, C_3 = m^2) \) gives the Proca equation:

\[ m^2 A^\rho = \partial_\sigma(\partial^\sigma A^\rho - \partial^\rho A^\sigma). \quad (5.5) \]

For a photon, with spin 1 and mass = 0, this gives

\[ \partial_\sigma(\partial^\sigma A^\rho - \partial^\rho A^\sigma) = 0, \]

---

45 See Robinson, 167ff, for a much more complete (and laborious) derivation of this Lagrangian.
which is the inhomogeneous Maxwell equation without electric currents, justifying the choice of constants. This is commonly written in terms of the electromagnetic field tensor
\[
F^\sigma{}^\rho := \partial^\sigma A^\rho - \partial^\rho A^\sigma
\]  
(5.6)
which gives
\[
\partial_\sigma F^{\sigma\rho} = 0.
\]
The Lagrangian for massless spin 1 is then
\[
\mathcal{L}_{\text{Maxwell}} = -\frac{1}{2} (\partial^\mu A^\nu \partial_\mu A_\nu - \partial^\mu A^\nu \partial_\nu A_\mu) = \frac{1}{4} F^{\mu\nu} F_{\mu\nu}.
\]  
(5.7)
For a massive spin 1 field,
\[
\mathcal{L}_{\text{Proc}} = -\frac{1}{2} (\partial^\mu A^\nu \partial_\mu A_\nu - \partial^\mu A^\nu \partial_\nu A_\mu) + m^2 A_\mu A^\mu = \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + m^2 A_\mu A^\mu.
\]  
(5.8)

6. Field theory of free fields

Why fields? It’s all because of QM and SR.

6.1. Problems of QM with SR

Think about single-particle QM. Squeeze the single particle into a space smaller than its Compton wavelength, \(\hbar/m\). By the Uncertainty Principle, its momentum is then \(\gg m\), so pair production can take place ... and we no longer have a single particle. But classical wave mechanics cannot handle such a case of particle production or creation. The Schrödinger equation for an electron has no place for the creation of an electron-positron pair.

Speaking of the Schrödinger equation; it is obviously not Lorentz-invariant since it includes spatial derivatives quadratically but time derivatives linearly.

Using the unitary time-evolution operator \(e^{-iHt}\), it is possible to derive an amplitude for evolution from a state at \(t=0\) over a space-like interval, i.e., \(\delta x > \delta t\), and the result is not (quite) the zero it should be! This result is clearly not acceptable to SR. More simply, you can’t measure an event here and now and then measure an event light-years away but one second later -- if “one second later” even means anything.

In EM theory, we measure fields at a point and consider the propagation of waves through space. The notion of fields provides locality to the equations. Needing just this in QM, we suspect that the answer is to use fields. But since measurable quantities are given by operators in QM, it seems we need operator-valued fields. Hence, quantum field theory.

So let’s try to understand what quantum field theory (QFT) is all about and what the concept of the wave function of a field means. To start with, two basic facts to keep in mind:

- “The world is made of quantum fields.”
- QFT is part of quantum mechanics, not something separate.

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46 The ideas of the next three paragraphs are borrowed from B&L, chapter 8, 75-76; and Zee, 3-4.
47 This and the next section are based on Sean Carroll, “The biggest ideas in the universe: fields”, http://www.preposterousuniverse.com/blog/2020/05/19/the-biggest-ideas-in-the-universe-9-fields/.
6.2. The wave function of a field

The wave function of a quantum field is like neither a particle wave function nor a classical field. The first is a function in space of one or many particles; the second, an entity with its own equations of motion. A quantum field has a wave function which is not a function of \( x \) but of the field configuration in all of space. In other terms, it is a function of a function, sometimes called a functional,

\[
\Psi[\phi(x)],
\]

where

\[
|\Psi[\phi(x)]|^2
\]

is the probability of finding the field in configuration \( \phi(\vec{x}) \), where \( \vec{x} \) ranges over all space for this configuration. Conceptually, \( \Psi \) measures the probability of simultaneously measuring the value of \( \phi \) throughout space, even though we cannot do this in reality. A classical scalar field is an object which maps real space (\( \mathbb{R}^3 \)) to a real number (\( \mathbb{R} \)); the quantum wave function of the field maps space to a complex number (\( \mathbb{C} \)), which allows us to calculate the probability that the entire field is in this configuration.

Ok, but how to get particles out of all these fields? Let's consider the simplest case, non-interacting fields, or "free" fields. Just as in one dimension we can use Fourier transforms to write any function as a sum of sine waves, so in 3-d, we can write the spatial field configuration as a sum of plane waves. Call these modes, each of which will be specified by a value of \( \vec{k} \), the wave number, and the "height", or amplitude, of the wave. Remember that the transform integrates (or sums) over all values of position, so the modes no longer depend on that. In other words, we have passed from a position (\( x \)) representation to a momentum (\( \vec{k} \)) one.

Now there are three sorts of energy for each mode:

- kinetic energy = \( \frac{1}{2} (\text{rate of change of field over time})^2 \)
- **gradient energy** = \( \frac{1}{2} (\text{rate of change of field over space})^2 \)
- potential energy = \( \frac{1}{2} m (\text{field value})^2 \)

These are really energy densities in QFT. The new, non-classical term, the gradient energy, as in (6.31), is due to the fact that the argument of the wave function, the field, varies over space as well as time. It can be considered a shear term. Of course, kinetic and gradient energy can be taken together as a 4-d energy term representing rate of change over spacetime. Changing the height, \( h \), of the plane waves will change all three energies as the square of \( h \), which is the same behavior as that of a simple harmonic oscillator (SHO). The system of a single mode can therefore be thought of as a SHO and that is a Good Thing, because we know the solutions to the Schrödinger equation for a SHO.

So the wave function of a complex field, which is a function of position throughout all of space, can be broken down via a Fourier transform into modes corresponding each to a single value of momentum (\( \vec{k} \)), where each mode depends on a single variable, \( h \), and no longer on position \( \vec{x} \). We can consider each mode as being fixed by the value of \( \vec{k} \) and being a function of the height:

\[
f_{\vec{k}}(h).
\]

We have thus replaced a complicated problem by a sum of simple ones. These will include the solutions to the equations of motion for free particles of spin 0, ½ and 1.

Now we can use all the tricks we learned about ladder and number operators for SHOs in QM, and these lead us to an interpretation. Energy eigenvalues are equally spaced, being given by

\[
E_n = \left(n + \frac{1}{2}\right)\hbar\omega,
\]

(6.1)
so by analogy, we may consider each level $n$ as corresponding to a wave function $\Psi_n$ of $n$ identical particles, each one of energy $\hbar \omega$. Then $\Psi_0$ is the wave function for no particles at all, the vacuum. Thus,

QFT can be interpreted as a theory of particles, because the discrete set of solutions to the Schrödinger equation$^{48}$ with correspondingly discrete equally-spaced energies can be thought of as states of different numbers of particles.

When we do this with the EM field, the energy states are called photons.$^{49}$ Now we can view the SHO ladder operators as creation and annihilation operators. We will be seeing this statement again. We will also see how the $\frac{1}{2} \hbar \omega$ terms can be considered the non-zero (!) value of the energy of the vacuum.

We have been considering non-interacting fields, without any interactions between particles. But it is impossible to observe particles without using interactions in order to do so. Also, interactions between particles are local in space. At $x = \pm \infty$, any interaction term goes to zero.

Nutshell: “One quantum field can be thought of as a superposition of different numbers of particles.” So we can “invent” QFT either by starting with particles or with fields.

One more time:

Free-particle fields can be expressed as sums of SHOs, each of momentum $\hbar \omega$. We can therefore use all the paraphernalia of diagonal and ladder operators to find eigenvalues of energy, which we can then interpret not just as quanta of energy, but as particles.

So much for an introduction to the idea; now on to the nitty-gritty.

6.3. Particles and fields and field theory

Quantum Lagrangians may be constructed from symmetry considerations, as we have done in preceding paragraphs. They may also be constructed by analogy with classical ones through the process of quantization, such as by replacing momentum, $p_x$, by the operator $-i\hbar \partial_x$.

Leonard Susskind cites four observed principles of physical laws.$^{50}$

- The action principle, which depends on the Lagrangian, without which there is no conservation of energy or momentum or the relationship between them and symmetry.$^{51}$
- Locality, which means that an event at one place can immediately affect only nearby places. This requires continuity of the Lagrangian and fields, which in turn says that the Lagrangian depends on the field and its first derivatives with respect to position.
- Lorentz invariance, so that the equations of motion are the same in every Lorentz-related reference frame. This is guaranteed if the Lagrangian is a scalar.
- Gauge invariance, such as for the EM field, where there are many equivalent descriptions of a system differing by arbitrary functions of position.

Quantization of classical Lagrangians must take these principles into account.

The Euler-Lagrange equations for a particle may be resumed as

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) = \frac{\partial L}{\partial q_i}. \quad (6.2)$$

$^{48}$ Or the Euler-Lagrange equations based on the Lagrangian.
$^{49}$ Lancaster and Blundell, 27.
$^{50}$ Susskind and Friedman (2017), 256-259.
$^{51}$ Susskind and Friedman (2017), 331.
In these equations, the coordinates \( q_i \) are the spatial location of the particle and so represent degrees of freedom of the particle. In general, it is moving, so the \( q_i \) are functions of time and \( \dot{q}_i \) is non-zero.

For a field, the variables of the equation are the components of the field. The \( q \)'s just designate the place where the field, \( \phi = \phi(q_i, t) \), is evaluated and are independent of time. We can then write one possible simple Lagrangian density which satisfies the four principles as follows.\(^{52}\)

\[
\mathcal{L} = -\frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - U(\phi). \tag{6.3}
\]

This is often written with a notation – difficult to see for older physicists\(^{53}\) – as

\[
\mathcal{L} = -\frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - U(\phi). \tag{6.4}
\]

Although the Lagrangian of a particle field is only a function of the \( q_i \) and \( \dot{q}_i \), i.e., the time derivative, in quantum field theory it is also a function of the \( \phi, \partial_\mu \phi \) including the space derivatives. The Euler-Lagrange equations for a field taking into account the spatial variations of the field can be calculated using differentiation by parts\(^{54}\) as

\[
\frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial \phi_{\mu}} = \frac{\partial \mathcal{L}}{\partial \phi}, \tag{6.5}
\]

where \( \phi \) is the value of the field at a given location. Applying the Euler-Lagrange equations to equation (6.3) with \( U(\phi) = \frac{1}{2} m^2 \phi^2 \) leads to the Klein-Gordon equation

\[
\left( \partial_\mu \partial^{\mu} + m^2 \right) \phi = 0, \tag{6.6}
\]

of which more later. We could also have derived this equation by using the classic dispersion relations

\[
p\mu p_\mu - m^2 = 0
\]

and substituting the standard momentum operator \( p_\mu \to i\hbar \partial_\mu \), letting the result act on a field as in (5.2).

A field variable \( \phi(t, x) \) corresponds to a degree of freedom of the system at each value of its independent variables, even if these are infinite in number.\(^{55}\) The \( \phi \) are functions of time whereas the \( q \)'s are neither functions of time nor degrees of freedom. The \( \phi(t, x) \) are functions of both time and position, so in general a field Lagrangian depends not only on \( \phi(t, x) \) but also on \( \dot{\phi}(t, x) \) and \( \partial_x \phi(t, x) \). Hence the extra terms in (6.5).

Then the action is as follows.

\[
\text{Action} = \int dt L(\phi, \dot{\phi}, \partial_x \phi) = \int dt \int d^3 x \mathcal{L}(\phi, \dot{\phi}, \partial_x \phi), \tag{6.7}
\]

where \( \mathcal{L} \) is the Lagrangian density. The conjugate momentum density of the field \( \phi \) is

\[
\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \tag{6.8}
\]

and the Hamiltonian is given by

\[
\mathcal{H} = \pi \dot{\phi} - \mathcal{L}. \tag{6.9}
\]

So in order to pass from particle mechanics to field mechanics, we make the following correspondences:

\[^{52}\text{Susskind and Friedman (2017), 336.}\]
\[^{53}\text{Although still not as bad as the horrible dot notation for left and right-chiral spinors.}\]
\[^{54}\text{Klauber, 18-19.}\]
\[^{55}\text{Susskind and Friedman (2017), 363-366.}\]
• \( t \to x^0, x^i(t) \to \phi(x^i) \), so \( x^\mu \to \phi(x^\mu) \)
• \( L, H \to \mathcal{L}, \mathcal{H} = \) Lagrangian and Hamiltonian densities
• The Euler-Lagrange equations, the definition of the Hamiltonian and the action are those of (6.5), (6.9) and (6.7).

6.4. EM field Lagrangian and Maxwell’s equations

For the EM field, making the Lagrangian Lorentz invariant is accomplished by constructing it from Lorentz-invariant quantities, scalars built from the components of the Lorentz-covariant EM field tensor \( F_{\mu\nu} \), defined in terms of the 4-d vector potential in equation (5.6). The Lagrangian without charges or currents then is usually written

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}.
\] (6.10)

Doing the sums and remembering minus signs from raising or lowering indices, \( \mathcal{L} \) reduces to

\[
\mathcal{L} = \frac{1}{2} (E^2 - B^2).
\] (6.11)

The factors of \( \frac{1}{2} \) or \( \frac{1}{4} \) are conventions which are without physical importance. Substituting the vector potential components from (5.6) leads to the Lagrangian for the EM field

\[
\mathcal{L} = -\frac{1}{4} (A_{\nu,\mu} - A_{\mu,\nu}) (A^{\nu,\mu} - A^{\mu,\nu}).
\] (6.12)

This is the equation for no charges or current. If such are present, we must add a term so that the Lagrangian for the EM field with currents is

\[
\mathcal{L} = -\frac{1}{4} (A_{\nu,\mu} - A_{\mu,\nu}) (A^{\nu,\mu} - A^{\mu,\nu}) + J^\mu(x) A_\mu(x),
\] (6.13)

where

\[
J^\mu = (\rho, J^i),
\] (6.14)

the charge density-current four-vector. From this Lagrangian, the Euler-Lagrange equations, which in this context are

\[
\frac{\partial}{\partial X^\nu} \frac{\partial \mathcal{L}}{\partial (A_{\mu,\nu})} = \frac{\partial \mathcal{L}}{\partial A_\mu},
\]

may be used to derive Maxwell’s equations starting with the vector potential in the EM field tensor and the requirement of gauge invariance, which in turn requires that the current 4-vector obey the continuity equation

\[
\partial_\mu J^\mu = 0.
\] (6.15)

6.5. The Schrödinger and Heisenberg pictures

The classic view of QM is the so-called Schrödinger picture. The time-dependent Schrödinger equation in the case where the Hamiltonian is independent of time leads to the unitary time-evolution operator \( \hat{U}(t) \) such that

\[
|\Psi(t)\rangle = \hat{U}(t)|\Psi(0)\rangle
\]

56 Susskind and Friedman (2017) 339-341.
57 Susskind, 341-346. The factor of \( \frac{1}{4} \) is a convention, according to Susskind.
The expectation value of a Hermitian operator $L$ in terms of the time-evolution operator then can be written as

$$\langle L \rangle = \langle \Psi(t) | L | \Psi(t) \rangle = \langle \Psi(0) U^\dagger(t) | L | U(t) \Psi(0) \rangle,$$

which we can group in two ways

$$\langle L \rangle = \langle \Psi(0) U^\dagger(t) | [L] | U(t) \Psi(0) \rangle = \langle \Psi^S(t) | L^S | \Psi^S(t) \rangle \quad (6.16)$$

or

$$\langle L \rangle = \langle \Psi(0) [U^\dagger(t) | L | U(t)] | \Psi(0) \rangle = \langle \Psi^H(t) | L^H(t) | \Psi^H \rangle, \quad (6.17)$$

both being equivalent in result. But the grouping of (6.16) represents a constant operator in the context of changing wave functions, whereas (6.17) represents a time-dependent operator acting on constant wave functions. The former, to which we are most used, is called the Schrödinger picture of QM (denoted by a superscript S); the latter, the Heisenberg picture (superscript H). The Heisenberg picture is the one most used in field theory.

To go from the Schrödinger to the Heisenberg picture,

- replace $\Psi(t)$ by $\bar{\Psi}(0)$, and
- define the Heisenberg operator by

$$L^H(t) = U^\dagger(t) L^S U(t). \quad (6.18)$$

The first of these two steps means

$$\Psi^S(t) = U(t) \Psi(0) = U(t) \Psi^H(t),$$

so

$$\Psi^H = U^\dagger(t) \Psi^S. \quad (6.19)$$

Differentiate by parts and use $dL^S/dt = 0$ to get

$$\frac{dL^H(t)}{dt} = -\frac{i}{\hbar} [L^H(t), H], \quad (6.20)$$

where $H$ in the commutator means the Hamiltonian. This is the Heisenberg equation of motion. In classical mechanics in terms of Poisson brackets, $\{,\}$,

$$\frac{du}{dt} = \frac{\partial u}{\partial q_i} \dot{q}_i + \frac{\partial u}{\partial p_i} \dot{p}_i + \frac{\partial u}{\partial t} = \{u, H\} + \frac{\partial u}{\partial t}, \quad (6.21)$$

where the last term is usually zero. Comparison of these two equation suggests that part of quantization will be the replacement

$$\{u, H\} \rightarrow -\frac{i}{\hbar} [u, H]. \quad (6.22)$$

Incidentally, the expectation value for any quantum field – scalar, spinor or vector – is zero, so fields themselves are unmeasurable. Fortunately, this is not the case with operators such as energy, momentum or charge.\(^{58}\)

---

\(^{58}\) Klauber, 190. He says there are exceptions “… more advanced areas of QFT.”
6.6. Quantization

There are two methods of introducing QFT, each being useful according to the circumstances.

- The more historical formulation is **canonical quantization**, which seeks to quantify a classical theory while maintaining its structure, particularly its symmetries. It is based on the correspondence (6.22) between classical Poisson brackets and QM commutations.

- The more modern method is the **path integral** formulation, which is a generalization of the action principle of classical mechanics, based on Feynman's notion of a sum over all possible paths.

One method may be more useful than the other depending on the situation studied. We will consider the path integral formulation in section 9.

In brief, the **canonical quantization** from classical physics to quantum field theory goes through two conceptual steps of quantization of the Lagrangian:

1. The first step can be seen either as taking into account the Heisenberg uncertainty principle or as imposing commutation relations on certain dynamical quantities. Either one leads from classical to quantum mechanics by the **canonical commutation relation**:

   \[ [\hat{x}, \hat{p}] = i\hbar. \] (6.23)

2. Then the jump to QFT requires considering the fields not as states (variables) but as operators, which must themselves obey commutation relations (6.26).

The requirements of SR also must be taken into account and this can be done before or after quantization. In other words, “relativize” and “quantize” – in either order.

The first step in quantization comes from the equivalence between classical Poisson brackets,

\[ \{q^i, p_j\} = \delta^i_j, \quad \{q^i, q_j\} = \{p^i, p_j\} = 0, \] (6.24)

and QM commutator brackets, which suggests

\[ \{q^i, p_j\} = \delta^i_j \rightarrow [q^i, p_j] = i\hbar \delta^i_j. \] (6.25)

From one author to another, this may be called **first quantization** (or **particle quantization**) or **canonical quantization**. It leads from classical mechanics to NRQM and from equations for particles to equations for waves – probability waves. The result is still in terms of states in spacetime coordinates, not fields.

The second step is then to require that the fields, \( \phi \), now be operators.\(^{59}\) Then the so-called **second quantization** (or **field quantization**) requires

\[ [\phi(x, t), \phi(x', t')] = [\pi(x, t), \pi(x', t')] = 0 \]

\[ [\phi(x, t), \pi(x', t')] = i\hbar \delta(t - t') \delta(x - x'). \] (6.26)

We will see that second quantization leads from waves back to particles, as in section 6.2 And more...

It is worthwhile to remember that neither Newton nor Einstein derived their respective formulas for gravity from other theories, but rather by letting experimental results and general principles inform inspired guesswork.

We will follow the schema of Lancaster and Blundell, wherein **canonical quantization** is a series of steps for going from a classical field theory to a quantum one. They see the method as comprised five steps.\(^{60}\)

---

59 Klauber, 42.
60 Adapted from Lancaster and Blundell, 98ff.
I. Start by preparing a Lagrangian for a classical relativistic field, considered as complex. This step generally involves some creative guesswork – or symmetries. Write this as a Lagrangian density. It is then useful to employ the Euler-Lagrange equation to derive the equations of motion and find their solutions, which will be interpreted as fields.

II. Evaluate the momentum and Hamiltonian densities in terms of fields using (6.8) and (6.9). Find the energy in terms of number operators.

III. Make the fields quantum mechanical by imposing commutation relations (6.26) on them, thus treating the fields as operators. Enforcement of the commutation relations on the fields will lead to commutation relations for the amplitudes.

IV. Work out the fields in terms of creation and annihilation operators based on the amplitudes. These would allow creation, annihilation and, so, mutation of particles, the basic steps of particle interactions.

V. Use normal ordering (section 6.7.8) to avoid infinite energies.

As an example, consider a massive scalar field (meaning one with mass, not necessarily a very huge one). Note that we are using the metric signature

\((+,-,-,-)\).

Would that there were a standard (preferably, this one). There almost is, at least among field theorists. Table 2 tells it all.

<table>
<thead>
<tr>
<th>Author(s)</th>
<th>metric signature</th>
<th>most-used (\gamma) matrix basis</th>
</tr>
</thead>
<tbody>
<tr>
<td>Blundell &amp; Lancaster</td>
<td>+++-</td>
<td>chiral</td>
</tr>
<tr>
<td>Carroll</td>
<td>-+++</td>
<td></td>
</tr>
<tr>
<td>Griffiths</td>
<td>+++</td>
<td>Dirac</td>
</tr>
<tr>
<td>Jeevanjee</td>
<td>+++- (!)</td>
<td>NA</td>
</tr>
<tr>
<td>Klauber</td>
<td>+++</td>
<td>Dirac</td>
</tr>
<tr>
<td>Maggiore</td>
<td>+++</td>
<td>chiral</td>
</tr>
<tr>
<td>Peskin &amp; Schroeder</td>
<td>+++</td>
<td>chiral</td>
</tr>
<tr>
<td>Robinson</td>
<td>-+++</td>
<td>chiral</td>
</tr>
<tr>
<td>Schwartz</td>
<td>+++</td>
<td>chiral</td>
</tr>
<tr>
<td>Schwichtenberg</td>
<td>+++</td>
<td>chiral</td>
</tr>
<tr>
<td>Susskind(^{\text{61}})</td>
<td>-+++</td>
<td>NA</td>
</tr>
<tr>
<td>Zee</td>
<td>?</td>
<td>both</td>
</tr>
</tbody>
</table>

Table 2. Comparison of metric signatures and \(\gamma\) bases by author.

Among QFT specialists, \((+1,-1,-1,-1)\) does seem to be the standard (Sorry, Mr Robinson) – to the extent that there is one.

\(^{61}\) Susskind claims \((-1,1,1,1)\) is generally used in GR and he seems to be right. It is also used by Carroll, Schutz and Hartle, but not by Lambourne or Collier.
6.7. Relativistic scalar Lagrangian (Klein-Gordon)

We start with the classical version

6.7.1. Classical relativistic scalar Lagrangian

The standard classical relativistic Lagrangian density for a massive scalar field starts from equation (6.3) with a potential energy term as follows:

\[ \mathcal{L} = \frac{1}{2} [\partial_{\mu} \phi(x)]^2 - \frac{1}{2} m^2 [\phi(x)]^2]. \tag{6.27} \]

This form was initially proposed because it is a scalar composed of Lorentz-invariant four-vectors. Its ultimate adoption is due to the fact that it works.

Since this is the classical relativistic Lagrangian, the field is real. The Euler-Lagrange equations really are easy to solve for the equation of motion of this Lagrangian and the result is the **Klein-Gordon equation**: \(^{62}\)

\[(\partial_{\mu} \partial^\mu + m^2) \phi = 0, \tag{6.28}\]

with

\[ \mu^2 = \frac{m^2 c^2}{\hbar^2} = m^2 \]

in natural units \((c = \hbar = 1)\).

6.7.2. Complexification

In the above equation, \(\phi\) is a real, classical field, not a wave function. For QFT, we must consider the field as being **complex**, not real. As a result, we must distinguish between the field \(\phi\) and its complex conjugate \(\phi^\dagger\), so the **Lagrangian density** takes on the complex formula

\[ \mathcal{L} = \frac{1}{2} [(\partial_{\mu} \phi^\dagger \partial^\alpha \phi - m^2 \phi^\dagger \phi)] = \frac{1}{2} [(\phi^\dagger \partial \phi^\dagger - \nabla \phi^\dagger \cdot \nabla \phi - m^2 \phi^\dagger \phi)], \tag{6.29} \]

where the upper zero indicates spin (which we will see shortly) and the lower, a free particle.

We can calculate the **conjugate (canonical) momentum density**

\[ \pi^\mu(x) = \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \phi(x))} = \partial^\mu \phi(x), \tag{6.30} \]

whose timelike component is just \(\partial^0 \phi(x)\). Then the Hamiltonian density is

\[ \mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \mathcal{L} = \frac{\partial L}{\partial \dot{\phi}} + \frac{\partial L}{\partial \dot{\phi}^\dagger} \dot{\phi}^\dagger = \]

\[ \frac{1}{2} (\dot{\phi}^\dagger \dot{\phi} + \nabla \phi^\dagger \cdot \nabla \phi + \mu^2 \phi^\dagger \phi) = \frac{1}{2} (\partial_{\mu} \phi^\dagger \partial^\mu \phi + \mu^2 \phi^\dagger \phi). \tag{6.31} \]

where we distinguish between fields considered different for the sum, in this case \(\phi\) and \(\phi^\dagger\). Now we see that the energy has three components – kinetic energy of motion in time, what we have called gradient energy in section 6.2 and potential energy (the mass term). Because the two fields \(\phi\) and \(\phi^\dagger\) are considered separate, the Euler-

\(^{62}\) Klauber, 42.
Lagrange equation leads to a Klein-Gordon equation for each field: From here on, \( \phi \) represents a wave.

\[
(\partial_\mu \partial^\mu + \mu^2)\phi = 0 \tag{6.32a}
\]

\[
(\partial_\mu \partial^\mu + \mu^2)\phi^\dagger = 0 \tag{6.32b}
\]

Care must be taken not to confuse the \( \mu \) for mass and those in the subscripts and superscripts. Two different equations means two solutions, each of which can be expressed as a series of plane waves for the discrete case

\[
\phi(x) = \sum_k \frac{1}{\sqrt{2V\omega_k}}a(\vec{k})e^{-ikx} + \sum_k \frac{1}{\sqrt{2V\omega_k}}b(\vec{k})e^{ikx} := \phi^+ + \phi^- \tag{6.33a}
\]

\[
\phi^\dagger(x) = \sum_k \frac{1}{\sqrt{2V\omega_k}}b(\vec{k})e^{-ikx} + \sum_k \frac{1}{\sqrt{2V\omega_k}}a(\vec{k})e^{ikx} := \phi^{\dagger+} + \phi^{\dagger-} \tag{6.33b}
\]

or as an integral for the continuous case

\[
\phi(x) = \int \frac{d^3\vec{k}}{\sqrt{2(2\pi)^3\omega_k}}a(\vec{k})e^{-ikx} + \int \frac{d^3\vec{k}}{\sqrt{2(2\pi)^3\omega_k}}b(\vec{k})e^{ikx} := \phi^+ + \phi^- \tag{6.34a}
\]

\[
\phi^\dagger(x) = \int \frac{d^3\vec{k}}{\sqrt{2(2\pi)^3\omega_k}}b(\vec{k})e^{-ikx} + \int \frac{d^3\vec{k}}{\sqrt{2(2\pi)^3\omega_k}}a(\vec{k})e^{ikx} := \phi^{\dagger+} + \phi^{\dagger-} \tag{6.34b}
\]

Note that the second term in each solution is only found in solutions of the equations resulting from the Lagrangian (6.29) of relativistic quantum mechanics and not those of the Schrödinger equation. Because the energy operator

\[ \hat{E} = i\partial_t, \]

(with \( \hbar = 1 \)) solutions of these equations with \(-ikx = -i(\vec{E}t - \vec{p} \cdot \vec{x})\) in the exponent result in positive energies, but those with \(+ikx = i(\vec{E}t - \vec{p} \cdot \vec{x})\), in negative energies. So the distinction here is that the + terms, \(\phi^+\) and \(\phi^{\dagger+}\), correspond to RQM states of positive energy (terms proportional to \(e^{-ik \cdot x}\)) whereas the − terms, \(\phi^-\) and \(\phi^{\dagger-}\) correspond to states of negative energy. So \(\phi^{\dagger+}\) cannot be the complex conjugate of \(\phi^+\), but a different field altogether. In fact, \((\phi^-)^\dagger = \phi^{\dagger-}\). The + or − superscripts have nothing to do with whether they operate on particles or antiparticles. This is all exceedingly messy, but the message to take home about creation and annihilation operators will be derived soon as (6.40).

### 6.7.3. Convert fields to operators by imposing commutation relations

This is step III of second quantization. Do this by imposing the commutation relations of equations (6.26) on the fields and by writing them as operators (with hats on them). Assuming the same value for time, this gives field commutators

\[
[\hat{\phi}(\vec{x}, t), \hat{\pi}(\vec{y}, t)] = i\delta^{(3)}(\vec{x} - \vec{y}). \tag{6.35}
\]

Or, in 4-d,

\[
[\hat{\phi}(x), \hat{\pi}(y)] = i\delta^{(4)}(x - y). \tag{6.36}
\]

Starting with the commutation relation for fields (6.35), plug in equations (6.33) with the x system corresponding to \((\omega, \vec{k})\) and the y system to \((\omega', \vec{k}')\) and expand. The result shows that the coefficients, the a’s and b’s, are

---

63 Klauber, 49.  
64 Klauber, 50.  
themselves operators and must satisfy the commutation relations:

\[ [a(\vec{k}), a^\dagger(\vec{k}')] = [b(\vec{k}), b^\dagger(\vec{k}')] = \delta_{\vec{k}\vec{k}'} \quad \text{(discrete)}; \]

\[ = \delta(\vec{k} - \vec{k}') \quad \text{(continuous)}. \] (6.37)

From commutation relations for the fields, we have deduced commutation relations for the amplitudes, or coefficients. This depended on the form of the solutions for the fields as sums of modes (mode expansions).

### 6.7.4. From parameters to wave functions to fields to ...

Let’s pause a moment to consider just what is going on.

In classical mechanics, we start with either Newton’s laws or a Lagrangian in terms of coordinates and their derivatives (or momenta) and we can solve them for the behavior of the system in terms of these spacetime-dependant variables. Schrödinger quantized classical mechanics by making operators out of these quantities and this makes some of them non-commuting. As operators, they then needed something on which to operate, so the wave function (or state vector) was born. But what was that? With the Born rule, the absolute square of the wave function was seen to represent the probability of the system’s being in a certain state, in which the state variables may take on eigenvalues given by the momentum and position operators operating on the state vector.

So now we have a wave function. But with the invention of QFT, second quantization forced the quantization of the wave functions of the field by the same trick of turning them into operators and imposing commutation rules on them. So what do they operate on? Enter a new state vector. It’s a bit confusing here, because the Dirac bra-ket convention had already been used in ordinary QM, but it is used again to denote the “thingy” operated on by the field operators. These new QFT state vectors will be used to calculate expectation values of observables.

Since the fields are operators and evolve in time – by equations (6.33) or (6.34) – then we are in the Heisenberg picture. Discussions of the Dirac and Proca equations will also be in the HP. It can indeed be shown that the Klein-Gordon field equation (6.28) is equivalent to the Heisenberg equation of motion (6.20). We will see later, in section 8.1, that the time-evolution operator in the IP (interaction picture, a special case of the HP) is governed by the free part only of the Hamiltonian (or Lagrangian).

### 6.7.5. Number operators, Hamiltonian and energy

Once the amplitudes \( a(\vec{k}), a^\dagger(\vec{k}'), b(\vec{k}), b^\dagger(\vec{k}') \) are known to satisfy the above commutation relations, then we can plug the solutions (6.33) into the Hamiltonian density and calculate the Hamiltonian

\[ H = \int \mathcal{H} d^3x. \]

The result is the free-particle Hamiltonian

\[ H = \sum_k \omega_k \left( N_a(\vec{k}) + \frac{1}{2} + N_b(\vec{k}) + \frac{1}{2} \right), \] (6.38)

expressed in terms of number operators:

\[ N_a(\vec{k}) = a^\dagger(\vec{k}) a(\vec{k}); \quad N_b(\vec{k}) = b^\dagger(\vec{k}) b(\vec{k}). \] (6.39)

---

66 Klauber, 52-53.

67 Klauber, 79-80.

68 The following notation is from Klauber, 54-55.
As Klauber says, these two equations “lie at the heart of QFT...”\textsuperscript{69} Again, (6.38) refers to a free particle of spin 0. We shall see very soon that the a terms correspond to particles and the b terms to antiparticles.

In words, in an occupation number representation, $N_\alpha(\vec{k})$ is the number operator whose eigenvalue $n_\alpha(\vec{k})$ is the number of a particles with 3-momentum $\vec{k}$ in the state. The two extra factors of $\frac{1}{2}$ in (6.38) are considered to represent the zero-point energy (ZPE) or vacuum energy (VE). It has its origin here in the non-commutation of the fields and their conjugate momenta from second quantization. Although this is often interpreted as virtual particles popping in and out of existence, there is nothing in the equations for the number operators (which contain the time dependence in QFT) for any variation in the number, which however should vary constantly if virtual particles are ceaselessly created and destroyed. In addition, we have calculated (6.38) for a free field with no interactions with anything else, meaning no virtual particles are appearing or disappearing.\textsuperscript{70} At least, not as far as we know at this point.

The form of equation (6.38) shows that all energies in QFT are positive, even for the b particles, which are antiparticles, as we shall soon see.

### 6.7.6. Creation and annihilation operators

Using the commutation relations (6.37) and the number operators (6.39), one can show, as for a SHO that the a’s and b’s are creation and destruction operators as follows:

$$
a^\dagger(\vec{k})|n_\vec{k}\rangle = \sqrt{n_\vec{k} + 1}|n_\vec{k} + 1\rangle
$$

$$
a(\vec{k})|n_\vec{k}\rangle = \sqrt{n_\vec{k}}|n_\vec{k} - 1\rangle
$$

$$
b^\dagger(\vec{k})|\bar{n}_\vec{k}\rangle = \sqrt{\bar{n}_\vec{k} + 1}|\bar{n}_\vec{k} + 1\rangle
$$

$$
b(\vec{k})|\bar{n}_\vec{k}\rangle = \sqrt{\bar{n}_\vec{k}}|\bar{n}_\vec{k} - 1\rangle. \tag{6.40}
$$

This result also is due to second quantization, which has changed non-QFT constant coefficients into operators with commutation operators.\textsuperscript{71} This is clear from

$$
\hat{n}a^\dagger|n\rangle = \hat{a}^\dagger\hat{a}a^\dagger|n\rangle = (\hat{a}\hat{a}^\dagger + 1)a^\dagger|n\rangle = (n + 1)a^\dagger|n\rangle,
$$

where the third step is due to the commutation relation.

Note that use of these operators on a state gives a new state. So the operators do not have eigenvalues and do not represent observables. Furthermore, since the a’s and b’s are operators, then $\phi(x)$ and $\phi^\dagger(x)$, as defined in (6.33) and (6.34) must also be operators. Since these equations are mode expansions of SHO solutions, it should come as no surprise that we come upon number operators and ladder operators.

Consider again the solutions of equations (6.33) and (6.34). We will confirm in (6.45) that the a’s are particle operators, whereas the b’s are antiparticle operators. From equations (6.40), we see which are creation and which annihilation\textsuperscript{72} operators. So there are four cases.

---

\textsuperscript{69} Klauber, 54.

\textsuperscript{70} Klauber, 55-56.

\textsuperscript{71} Even if we have not always put the hat on the a and b operators.

\textsuperscript{72} Or should we say “terminator” to be more up to date? ;)

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Table 3. Particle/antiparticle creation/annihilation operators

<table>
<thead>
<tr>
<th>Field operator from equation (6.33) and (6.34)</th>
<th>Term contains</th>
<th>action</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi$</td>
<td>$\phi^+$</td>
<td>$ae^{-ikx}$ destroys particles</td>
</tr>
<tr>
<td></td>
<td>$\phi^-$</td>
<td>$b^* e^{ikx}$ creates antiparticles</td>
</tr>
<tr>
<td>$\phi[^+]$</td>
<td>$\phi[^+]$</td>
<td>$be^{-ikx}$ destroys antiparticles</td>
</tr>
<tr>
<td></td>
<td>$\phi[^-]$</td>
<td>$a^* e^{ikx}$ creates particles</td>
</tr>
</tbody>
</table>

So we have that
- $\phi$ is the **total lowering operator**, since it destroys particles and creates antiparticles; and
- $\phi[^+]$ is the **total raising operator**, since it destroys particles and creates antiparticles.

Putting in a minus sign for antiparticles, the **total particle-number operator** is then

$$N(\phi) = \sum_{\vec{k}} \left( N_a(\vec{k}) - N_b(\vec{k}) \right).$$

(6.41)

Now we can see why we need QFT. It can handle particle creation and annihilation and, so, particles’ changing into one another. But it can also – and mainly – handle multi-particle states.

### 6.7.7. Why second quantization?

First quantization produces discrete modes for one system, such as the hydrogen atom or a particle in a box. Second quantization produces integral numbers of excitations of each such mode.

As Schwartz says: “At the risk of oversimplifying things things a little, that [second quantization] is all there is to quantum field theory. The rest is just quantum mechanics.” In his analysis, QFT brings two great advantages:

- Because of the creation/annihilation operators, it can account for multi-particle states.
- The $n^{th}$ excitation state of a system is considered to represent $n$ particles.

For instance, applying a creation operator twice to the vacuum $|0\rangle$ creates a two-particle state, subject to the Exclusion Principle.

### 6.7.8. Normal ordering

If we use the Hamiltonian of equation (6.38), the terms $\frac{1}{2}$ when integrated over space will lead to an infinite energy, which most physicists find quite embarrassing. A way to avoid this problem is to use **normal ordering**, which means rearranging a string of operators so that creation operators are on the left and annihilation operators on the right, otherwise in the original order. This procedure supposedly corrects an ambiguity in the order of terms in the classical Lagrangian and effectively removes the infinities, which have their origin in the non-commutation of certain a’s and b’s. But since it pretends momentarily that the creation and annihilation

---

73 These are Klauber’s notation. Everybody else uses $\phi[^+]$ to mean creation and $\phi[^-]$ annihilation, as will we in future paragraphs. This will agree with $a[^+]$ and $b[^+]$ as creation and $a$ and $b$ as annihilation operators.

74 I admit to not understanding this, the second part. If negative exponent means positive energy, these are particles, not antiparticles. However, $b$ seems to prim, saying they are antiparticles...

75 Schwartz, 20.

76 Klauber, 60-61; Lancaster and Blundell, 104-105.
operators temporarily commute – which they do not – the procedure is complex. In fact, the non-commutation of the operators is fundamental elsewhere in QFT.

While normal ordering is questionable as a means of eliminating infinite vacuum energies, it can be legitimate as well as useful if one takes account accurately of the commutation relations for the operators exchanged. This is the subject of the use of Wick’s theorem of contractions (section 8.3).

6.7.9. Résumé

After the nitty-gritty, let’s look at an overview of what we have done and found out about quantization.

I. Start with a relativistic Lagrangian density for particles

\[ \mathcal{L} = \frac{1}{2} \left[ \partial_\mu \phi(x) \right]^2 - \frac{1}{2} m^2 [\phi(x)]^2 \]  

and make the fields, \( \phi \), complex:

\[ \mathcal{L} = \frac{1}{2} \left[ \left( \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi \right) \right] = \frac{1}{2} \left[ \left( \phi^\dagger \dot{\phi} - \nabla \phi^\dagger \cdot \nabla \phi - m^2 \phi^\dagger \phi \right) \right] \]  

Derive the canonical momentum

\[ \pi^\mu(x) = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi(x))} = \partial^\mu \phi(x) \]  

and use it to write the Hamiltonian density

\[ \mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \dot{\phi} - \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi} \phi + \frac{\partial \mathcal{L}}{\partial \phi^\dagger} \phi^\dagger = \frac{1}{2} \left( \phi^\dagger \dot{\phi} + \nabla \phi^\dagger \cdot \nabla \phi + \mu^2 \phi^\dagger \phi \right) = \frac{1}{2} \left( \partial_\mu \phi^\dagger \partial^\mu \phi + \mu^2 \phi^\dagger \phi \right). \]

Use this or the Euler-Lagrange equations to write the Klein-Gordon equation (wave equation, equivalent to the Schrödinger equation)

\[ (\partial_\mu \partial^\mu + \mu^2) \phi = 0 \]  

(6.32a)

\[ (\partial_\mu \partial^\mu + \mu^2) \phi^\dagger = 0 \]  

(6.32b).

Then solve these for the plane-wave solutions

\[ \phi(x) = \sum_k \frac{1}{\sqrt{2V \omega_k}} a(\vec{k}) e^{ikx} + \sum_k \frac{1}{\sqrt{2V \omega_k}} b(\vec{k}) e^{ikx} : = \phi^+ + \phi^- \]  

(6.33a)

\[ \phi^\dagger(x) = \sum_k \frac{1}{\sqrt{2V \omega_k}} b^\dagger(\vec{k}) e^{-ikx} + \sum_k \frac{1}{\sqrt{2V \omega_k}} a^\dagger(\vec{k}) e^{ikx} : = \phi^{+\dagger} + \phi^{-\dagger} \]  

(6.33b)

or the continuous solutions of equations (6.34). We have now done first quantization, converting a particle equation to a wave equation and so passing from classical to quantum mechanics. We now deal with wave fields.

II. Make the fields quantum mechanical by imposing the commutation relations

\[ \{ \phi^i(x, t), \pi_j(y, t) \} = \delta^i_j \delta(x - y) \rightarrow \{ [\phi^i(x, t), \pi_j(y, t)] = i\hbar \delta^i_j \delta(x - y); [\phi^r, \phi^s] = [\pi_r, \pi_s] = 0 \]  

(6.26)

on the fields \( \phi \) of the plane-wave solutions (6.33). This requirement leads to commutation relations on the
amplitudes:

\[ [a(\vec{k}), a^\dagger(\vec{k}')] = \delta(\vec{k} - \vec{k}') \quad \text{(discrete)}; \]
\[ = \delta(\vec{k} - \vec{k}') \quad \text{(continuous)}. \]  

(6.37)

The commutation relations constitute 2\textsuperscript{nd} quantization, showing how waves can give rise to particles.

III. Use these results to evaluate the Hamiltonian from the Hamiltonian density and find the energy states:

\[ H = \sum_k \omega_k \left( N_a(\vec{k}) + \frac{1}{2} + N_b(\vec{k}) + \frac{1}{2} \right). \]  

(6.38)

which look like those of a SHO in terms of a number operator

\[ N_a(\vec{k}) = a^\dagger(\vec{k})a(\vec{k}); \quad N_b(\vec{k}) = b^\dagger(\vec{k})b(\vec{k}). \]  

(6.39)

IV. Show that the amplitude operators can be used as creation and annihilation operators of particles.

V. Use normal ordering. More on this in section 8.3.

The number operators represent observables and the \( a \)'s will be used in propagators and interaction theory.

6.7.10. **Klein-Gordon particles are bosons**

Looking at equations (6.40) for how creation operators raise the number of particles in a state, we see that, just as in the case of raising operators for the NRQM SHO, we can write any state (ignoring \( b \) for the moment) as

\[ |n\rangle = \left( \frac{\hat{a}^\dagger}{} \right)^n |0\rangle. \]

This equation says that multiple particles occupy the same state \( |n\rangle \) and so, according to the Fermi exclusion principle, Klein-Gordon particles must be **bosons**.

6.7.11. **Continuity, currents and negative energy**

Using the standard QM method of multiplying the Schrödinger equation by the complex conjugate of the wave function, subtracting the complex conjugate of that and doing some differentiating by parts, we can derive the probability current for the K-G equation in QFT.\(^77\)

\[ \rho = j^0 = i \left( \frac{\partial \phi}{\partial t} \phi^\dagger - \frac{\partial \phi^\dagger}{\partial t} \phi \right) \]  

(6.42)

and

\[ j^\mu = i \left( \phi^\mu \phi^\dagger - \phi^\dagger \phi^\mu \right) \]  

(6.43)

so that the continuity equation is then

\[ j_{\mu}^\mu = \partial_\mu j^\mu = 0. \]  

(6.44)

Now things get interesting. Insert the K-G solutions (6.33) into the probability density (6.42), then derive the effective probability density

\[ \tilde{\rho} = \langle \phi_1 \phi_2 \ldots | \rho | \phi_1 \phi_2 \ldots \rangle, \]

supposing that the bras and kets are eigenstates of \( \vec{k} \), i.e., energy-momentum. The somewhat surprising and

\(^{77}\) Klauber, 45-46.
The minus sign on the second term comes partly from that on the second term of equation (6.42) and partly from derivatives of the negative-energy b states. The term looks like negative probability as well as negative energy, but if we multiply this by the appropriate charge q, then the second term makes perfect sense as charge density if the b particles are interpreted as antiparticles. This is therefore what is done.

We will say more about negative energies in section 6.8.2.

6.7.12. Comparison to a SHO

Similarities of K-G fields with a SHO are obvious and interesting, but no proof of QFT. Why? Well... 79

• Both SHOs and QFT have raising and lowering operators with similar behaviors on states.

However,

• SHOs are examples of bound states with a non-zero potential energy and hence subject to a force, whereas the K-G equation applies to free states with no potential or force.

• A SHO is a single particle with different energy states, whereas QFT can handle states of many particles in each energy state.

• A SHO represents a real particle oscillating in place with a non-sinusoidal wave function (Hermite polynomial), whereas QFT considers complex, sinusoidal waves which move through space.

So although SHOs may help interpreting or justifying QFT, they do not offer a rigorous derivation of it.

6.7.13. The Feynman propagator

Now we can see how the coefficient operators can be used to construct measurable quantities involving the creation and destruction of virtual particles. The Feynman propagator (or just propagator) is a mathematical representation of a virtual particle or antiparticle (such as a force-carrying boson) created at one point in space and time in the vacuum and destroyed at another point. Spoiler: We will use them to propagate a particle from vertex of a Feynman diagram to another. So far, we have only studied the Klein-Gordon equation for bosons, so for the moment we will stick to considering the propagator for a virtual boson.

Suppose a virtual particle is created at y and destroyed at a later time at x. This is physically indistinguishable from a virtual antiparticle of opposite charge going from x to y. In terms of the K-G solutions (6.34), the creation is represented by $\phi(y)\phi(y)$ and is followed by an annihilation given by $\phi(x)$, the two forming a time-ordered operator

$$\mathcal{T}\{\phi(x)\phi(y)\} = \phi(x)\phi(y), \quad \text{for} \quad t_y < t_x.$$ (6.46)

The time-ordering operator, $\mathcal{T}$, not really an operator, just says to put the earliest term on the right. Now we could instead suppose that $\phi(x)$ creates an antiparticle which is later annihilated by $\phi(y)$. This antiparticle creation-annihilation would be represented by

$$\mathcal{T}\{\phi(x)\phi(y)\} = \phi(y)\phi(x), \quad \text{for} \quad t_x < t_y.$$ (6.46)

In the vacuum state, the transition amplitude for the particle process is given by its vacuum expectation value

---

78 Klauber, 69.
79 These criteria are after Klauber, 69-70.
This reduces to a product of scalars representing the creation and destruction probability amplitudes, respectively, of the virtual particle. We define the Feynman propagator by

$$i\Delta_F(x - y) = \langle 0 | T \{ \phi(x) \phi^\dagger(y) \} | 0 \rangle.$$  

(6.48)

Using the solution expansions (6.34), this can be expressed in terms of commutators of the separate amplitudes of the K-G solutions

$$i\Delta_F(x - y) = \langle 0 | [\phi^+(x), \phi^-(y)] | 0 \rangle$$

for a virtual particle, and

$$i\Delta_F(x - y) = \langle 0 | [\phi^{1+}(y), \phi^{-}(x)] | 0 \rangle$$

for an antiparticle. Now plug in the K-G solutions (6.34) and use the results of the commutators to get an integral, which can then be converted to a contour integral and then back to the final integral

$$i\Delta_F(x - y) = \frac{i}{(2\pi)^4} \int_{-\infty}^{\infty} e^{-ik(x-y)} \frac{e^{-ik^2}}{k^2 - \mu^2 + i\epsilon} d^4k$$  

(6.49)

where $\epsilon \to 0$ after doing the integral. This is the Feynman propagator for spin 0 bosons. It is particularly "simple" in the momentum representation:

$$\Delta_F(k) = \frac{1}{k^2 - \mu^2 + i\epsilon}.$$  

(6.50)

So from the K-G equation and 2nd quantization, we can calculate a probability amplitude for the creation of virtual particles, a step towards writing the equation for an interaction. Remember, we assume free particles and so no interactions.

The good news is that, at least according to Klauber, all this makes up "... most of the basic principles in QFT."

## 6.8. Spin-½ particles – the Dirac equation

We saw how to find the Dirac Lagrangian from symmetry considerations, complete with $\gamma$ matrices, in (5.2).

$$\mathcal{L} = \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi,$$  

(6.51)

where the adjoint $\bar{\psi} = \psi^\dagger \gamma^0$. We can then use it to derive the Dirac equation.

Alternatively, the Dirac equation can be derived by starting with the QM dispersion relation

$$E^2 = m^2 + p^2 \rightarrow \partial^2 - m^2.$$  

We would like to split this up into two parts using something like $\sqrt{\partial^2 \pm im}$, but the square root of $\partial^2$ does not mean anything. The historic discovery of the appropriate equation lends some physical understanding. Dirac wanted such a linear equation also to be a solution to the Klein-Gordon equation, and so he considered

$$\nu^\mu \psi_\mu + m\psi,$$

but then $\nu^\mu$ would define a vector with a specific direction in space and so would not be Lorentz invariant. So the coefficients could not constitute a simple spacetime constant, but must be operators (matrices) of some sort. So

---

80 Klauber, 73. Not everybody puts in the factor 1 on the left-hand side or uses the $\Delta$ symbol.

81 Details of the calculation are in Klauber, 70-77. Yes, over 8 pages.
he tried

\[ (i\gamma^\mu \partial_\mu - m)\psi = 0. \]

In order to compare this with a dispersion relation, one can multiply it by the same quantity with a + sign, the result being

\[ -(\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu + m^2)\psi = 0, \]

which can be rewritten as

\[ \frac{1}{2}((\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu + \gamma^\mu \gamma^\nu \partial_\mu \partial_\nu) + m^2)\psi = 0. \]

Exchanging \( \mu \) and \( \nu \) in the second term leads to the anticommutator, so

\[ \left( \frac{1}{2} \{ \gamma^\mu, \gamma^\nu \} \right) \partial_\mu \partial_\nu + m^2)\psi = 0. \]

If the \( \gamma \) satisfy the condition for a Clifford algebra (4.31)\textsuperscript{82},

\[ \{ \gamma^\mu, \gamma^\nu \} = 2\eta^{\mu\nu}I, \tag{6.52} \]

with \( \eta \) being the Minkowski metric, the result is the desired Klein-Gordon equation,

\[ (\delta^2 + m^2)\psi = 0. \]

The Clifford algebra condition (6.52) tells us that the \( \gamma^\mu \) must be 4x4 matrices like the metric. The Clifford algebra requirement is satisfied if the \( \gamma \) matrices obey

\[ (\gamma^0)^2 = 1, \quad (\gamma^1)^2 = (\gamma^2)^2 = (\gamma^3)^2 = -1 \]

with the anticommutation relations

\[ \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu := \{ \gamma^\mu, \gamma^\nu \} = 0, \mu \neq \nu. \]

All this leads to the Dirac equation, the RQM wave equation for spin-\( \frac{1}{2} \) particles:

\[ (i\gamma^\mu \delta_\mu - m)\psi = 0. \tag{6.53} \]

One choice for the Dirac matrices, or gamma matrices, is the chiral representation or basis (or the Weyl basis):

\[ \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{6.54} \]

where the \( \sigma^i \) are our old friends the Pauli spin matrices

\[ \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

Since the Pauli matrices are 2x2 in dimension, the gammas are 4x4. In all their glory,

\[ \gamma^0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \]

\[ \text{N.B. The sign of the right-hand side depends on the metric signature. We use (+---). Robinson, for instance, uses the opposite. See Table 2.} \]
The notation becomes much simpler if we define
\[ \sigma^\mu = (I, \bar{\sigma}) \quad \text{and} \quad \bar{\sigma}^\mu = (I, -\bar{\sigma}). \] (6.56)
(Note the almost-invisible bar over the second quantity, \( \bar{\sigma}^\mu \).) Then we can write the chiral basis as
\[ \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}. \] (6.57)
Alternatively, one can use the mass or standard or Dirac representation or basis\(^{83}\):
\[ \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]
\[ \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \]
which is the same as the chiral basis except for \( \gamma^0 \).

The important point is:

**Physical results are independent of the choice of basis for the gamma vectors.**\(^{84}\)

### 6.8.1. Spin, helicity and chirality

It is informative to see how the system works in the chiral basis. We represent the operators as 2x2 matrices, knowing this to be a shorthand for 4x4 matrices, so they must operate on four-component vectors. In fact, the vectors are Dirac spinors, four-component spinors each composed of a pair of two-component Weyl spinors.

Let’s spell this out completely one time:

\[
\psi(x) = \begin{pmatrix} \psi_L(x) \\ \psi_R(x) \end{pmatrix} = \begin{pmatrix} \psi_{L1}(x) \\ \psi_{L2}(x) \\ \psi_{R1}(x) \\ \psi_{R2}(x) \end{pmatrix} = \begin{pmatrix} u_L \\ u_R \end{pmatrix} e^{-ip \cdot x} = \begin{pmatrix} u_{L1} \\ u_{L2} \\ u_{R1} \\ u_{R2} \end{pmatrix} e^{-ip \cdot x},
\]

the last two terms being for spinors. The subscripts will be explained below.

Using (6.57) and with \( p \cdot x = \hat{p}^0 t - \hat{p} \cdot \hat{x} \), the Dirac equation (6.53) gives

\[
\begin{pmatrix} 0 & \hat{p}^0\\ \hat{p}^0 & 0 \end{pmatrix} - \begin{pmatrix} \sigma \cdot \hat{p} \\ 0 \end{pmatrix} - \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \end{pmatrix} u(p) e^{-ip \cdot x} = 0,
\]

or

\[
\begin{pmatrix} -m & \hat{p}^0 - \sigma \cdot \hat{p} \\ \hat{p}^0 + \bar{\sigma} \cdot \hat{p} & -m \end{pmatrix} u(p) = 0,
\] (6.58)

where we consider only the spinor part of the solution. Remember that \( \sigma \) is a 2x2 matrix and \( m \) is supposed multiplied by a unit matrix. Let’s make the usual supposition that the momentum is along the z-axis. Then what

---

83 Griffiths calls it the ‘Bjorken and Drell’ convention. This is the one also used by Klauber. Schwichtenberg uses both this and the chiral representation, which is also used by Lancaster and Blundell. Actually these two are handy on different occasions.

84 Robinson, 183.
this equation represents in all is:
\[
\begin{pmatrix}
-m & 0 & p^0 - p^3 & 0 \\
0 & -m & 0 & p^0 + p^3 \\
p^0 + p^3 & 0 & -m & 0 \\
0 & p^0 - p^3 & 0 & -m
\end{pmatrix}
\begin{pmatrix} u(p) \end{pmatrix}.
\]
(6.59)

For the special case of a massless particle, (6.58) reduces to
\[
(p^0 - \sigma \cdot \hat{p})\psi_R = 0 \quad \text{and} \quad (p^0 + \sigma \cdot \hat{p})\psi_L = 0.
\]
(6.60)
so the equations of the two spinors, \(\psi_L\) and \(\psi_R\), are separate and each evolves independently of the other. In particular, one will not change into the other. Although, we have not yet stated what R and L refer to, each represents a two-component spinor; \(\psi_L\) occupies the top two components of the four-component spinor field, \(\psi_R\) the two lower.

Starting with the Lorentz algebra of section 4.6.1 and (4.31) and using the chiral representation and the Clifford algebra, one can derive the equations for Lorentz rotations and boosts of the vector \(\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}\). The form of the boost for \(\psi_L\) indicates clearly that it is a left-chiral spinor field; similarly, \(\psi_R\) transforms like a right-chiral spinor. So in the chiral representation for massless particles, the two chiralities are well separated.

Now if we consider massive particles, (6.58) becomes
\[
(p^0 - \sigma \cdot \hat{p})\psi_R = m\psi_L, \quad \text{and} \quad (p^0 + \sigma \cdot \hat{p})\psi_L = m\psi_R.
\]
(6.61)
Now the two states are coupled (by the mass) and may oscillate one into the other. If \(\psi_L\) and \(\psi_R\) are eigenstates of momentum and energy, we may eliminate one of the spinors, say \(\psi_L\) to show
\[
(p^0 + \sigma \cdot \hat{p})(p^0 - \sigma \cdot \hat{p})\psi_R = m^2 \psi_R,
\]
which gives the desired dispersion relation
\[
(p^0)^2 - (\hat{p})^2 = m^2.
\]
Since we may take either of the square roots of this, we may get positive or negative energy
\[
E = \pm \sqrt{p^2 + m^2}.
\]
In the extreme relativistic limit of very high energies, we can ignore the mass. So either in this case or that of a massless particle, equation (6.58) reduces via (6.57) to the Weyl equations
\[
(p \cdot \sigma)\psi_R = 0 \quad \text{and} \quad (p \cdot \tilde{\sigma})\psi_L = 0.
\]
These two equations then lead to
\[
\hat{p} \cdot \tilde{\sigma} u_R = u_R \quad \text{and} \quad \hat{p} \cdot \tilde{\sigma} u_L = -u_L,
\]
where \(\hat{p}\) is a unit vector in the direction of the particle’s motion, so \(\hat{p} \cdot \tilde{\sigma}\) is a way of writing the helicity operator. These say that in the extreme relativistic limit the particle’s spin is aligned along its direction of motion. By definition, if the eigenvalue of the helicity operator \(\hat{p} \cdot \tilde{\sigma}\) is +1, the spinor is considered to be right-handed (right-chiral); for eigenvalue -1, left-handed.

By convention, the antiparticle solutions of the Dirac equation have negative values of energy. In this case, the equations (6.60) for a massless antiparticle become

---

85 Robinson, 175-177.
86 Robinson, 185-6.
\(-|\vec{p}^0| - \sigma \cdot \hat{\vec{p}}\)\psi_R = 0 \quad \text{and} \quad \(-|\vec{p}^0| + \sigma \cdot \hat{\vec{p}}\)\psi_L = 0,

so that

\[\frac{\sigma \cdot \hat{\vec{p}}}{|\vec{p}|} \psi_R = -\psi_R \quad \text{and} \quad \frac{\sigma \cdot \hat{\vec{p}}}{|\vec{p}|} \psi_L = \psi_L,\]

for antiparticles.

The signs of the helicity are inverted from those for positive-energy states.

Similarly, the **chirality operator** is defined by

\[
\gamma^5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3 \quad (6.63)
\]

\[
= \begin{pmatrix}
-1 & 0 \\
0 & 1
\end{pmatrix}, \quad \text{only in the chiral basis.} \quad (6.64)
\]

The matrix \(\gamma^5\) can be shown to be invariant under Lorentz transformations. Its eigenstates are

\[
\begin{pmatrix}
\psi_L \\
0
\end{pmatrix}
\quad \text{and} \quad \begin{pmatrix}
0 \\
\psi_R
\end{pmatrix}
\]

with eigenvalues -1 and +1, respectively. The **projection operators**

\[P_\pm = \frac{1}{2} (1 \pm \gamma_5)\]

project out the left and right-chiral spinors:

\[P_+ \psi = \psi_R \quad \text{and} \quad P_- \psi = \psi_L.\]

### 6.8.2. Interlude – negative energies

Let’s be as clear as we can about how we handle seemingly-negative energies.

Suppose the Klein-Gordon equation has solutions like

\[\phi(x) = Ne^{-i\vec{p} \cdot \vec{x}} = Ne^{-i(Et - \vec{p} \cdot \vec{x})}.\]

then our convention is to take this to represent an incoming wave. Applying the momentum operator \(\hat{\vec{p}} = -i \partial_x\) returns the momentum \(p^x > 0\) (and \(E > 0\), by \(i\partial_t\)). So our convention supposes the wave comes in from the left, moving in the direction of increasing \(x\). A positive exponential term would give \(p^x < 0\), representing a particle moving in the other direction, i.e., either a particle outgoing or an incoming antiparticle, as we will now see.

A charged particle in an EM field obeys the classical equation\(^{87}\)

\[m \frac{d^2}{dx^2} x = q F_{\mu}^\nu \frac{dx^\mu}{d\tau}.\]

If we invert \(x \rightarrow -x\) the equation remains unchanged if we also invert the sign \(q\) of the particle. So a charged particle traveling backwards in time is equivalent to the oppositely charged particle moving forwards in time.

The probability-current density for a K-G particle is

\[J^\mu = (+q)2|N|^2 p^\mu = (+q)2|N|^2 (E, \vec{p})\]

for a particle with positive energy, where \(N\) is a normalization constant. If its energy is negative

\[J^\mu = (+q)2|N|^2 (-|E|, \vec{p}) = (-q)2|N|^2 (|E|, -\vec{p}).\]

\(^{87}\) This paragraph based on Blundell and Lancaster, 62-63. Theirs is the only book I know which explains this so clearly.
So we can handle a negatively-charged particle by making it an antiparticle (i.e., reversing its charge), considering its energy to be positive and reversing the sign of its three-momentum.

\[
\begin{pmatrix}
-|E| \\
\vec{p} \\
q
\end{pmatrix} \rightarrow \begin{pmatrix}
|E| \\
-\vec{p} \\
-q
\end{pmatrix}.
\]

(6.65)

Negative-energy particles are interpreted as positive-energy antiparticles moving in the opposite direction to that of the corresponding particle. The K-G equation with a specific energy then has two solutions:

\[
\phi(x) = \begin{bmatrix}
\text{incoming positive-energy particle} \\
\propto e^{-i(Et-\vec{p} \cdot \vec{x})}
\end{bmatrix} + \begin{bmatrix}
\text{outgoing negative-energy particle} \\
\propto e^{i(Et-\vec{p} \cdot \vec{x})}
\end{bmatrix}.
\]

Note that this discussion concerns fields considered to have energy -- positive or negative. In QFT, these “fields” will become operators, which don’t have energy. A state is the result of acting on the vacuum with a creation or annihilation operator and this will create either a state with positive energy or annihilate the state to nothing.

### 6.8.3. Solutions to the Dirac equation

Let’s go ahead and look at the solutions. For now, we will use the chiral, or Weyl, basis. Knowing that solutions of the Dirac equation must also be solutions of the Klein-Gordon equation, we start with the general form of these from (6.33) in terms of plane waves:

\[
\psi(x) \simeq u(p)e^{-ipx} + v(p)e^{ipx},
\]

(6.66)

where the first term represents solutions for positive-frequency (energy) particles and the second, for negative-frequency, since (reminder: metric signature +---)

\[
\hat{E}\psi(x) = i\partial_t \psi(x) = Eu(p)e^{-ipx} + (-E)v(p)e^{ipx}.
\]

Then (6.58) becomes two equations, one for positive frequency and one for negative:

\[
\begin{pmatrix}
-m \\
p \cdot \vec{\sigma} \\
-m
\end{pmatrix} u(p) = 0 \quad \text{and} \quad \begin{pmatrix}
-m \\
-p \cdot \vec{\sigma} \\
-m
\end{pmatrix} v(p) = 0.
\]

(6.67)

In the rest frame, where \( p=(m,0,0,0) \), the solutions are

\[
u = \begin{pmatrix}
\xi \\
\bar{\xi}
\end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix}
\eta \\
-\bar{\eta}
\end{pmatrix}
\]

where \( \xi \) and \( \eta \) are any two-component spinors, e.g.

\[
\begin{pmatrix}
1 \\
0 \\
0 \\
1
\end{pmatrix}, \quad u_\uparrow = \sqrt{m} \begin{pmatrix}
0 \\
1 \\
1 \\
0
\end{pmatrix}, \quad v_\uparrow = \sqrt{m} \begin{pmatrix}
-1 \\
0 \\
0 \\
1
\end{pmatrix}, \quad v_\downarrow = \sqrt{m} \begin{pmatrix}
0 \\
1 \\
0 \\
-1
\end{pmatrix}.
\]

(6.68)

Putting the mass back, let’s make the usual simplification where we take the momentum to be along the z-axis and consider only the spinor part of the solution. Remember that \( \sigma \) is a 2x2 matrix and \( m \) is supposed multiplied by a unit matrix. Then the positive-frequency equation (6.67) leads to

---

88 Copied from Blundell and Lancaster, 63.
89 Method of Schwartz, 188-189.
The solutions in the $p^3$ frame can be found to be\(^{90}\)

\[
\begin{pmatrix}
-m & 0 & p^0 - p^2 & 0 \\
0 & -m & 0 & p^0 + p^3 \\
p^0 + p^3 & 0 & -m & 0 \\
0 & p^0 - p^3 & 0 & -m \\
\end{pmatrix} u(p) = 0. \quad (6.69)
\]

\[
\begin{pmatrix}
\sqrt{E - p^2} & 0 \\
0 & \sqrt{E + p^2} \\
0 & \sqrt{E - p^2} \\
\end{pmatrix} \xi \]

In a more condensed, almost cryptic, form\(^{6.71}\)

\[
u(p) = \left( \begin{array}{c}
\sqrt{p \cdot \sigma} \\
\sqrt{p \cdot \sigma} \\
\sqrt{p \cdot \sigma} \\
\end{array} \right) \xi.
\]

Don't miss the bar over the $\sigma$ in the lower part of each one. In the case of $p = (E, 0, 0, p^z)$,

\[
u^1 = \left( \begin{array}{c}
\sqrt{E - p^2} \\
0 \\
\sqrt{E + p^2} \\
0 \\
\end{array} \right), \quad u^2 = \left( \begin{array}{c}
0 \\
\sqrt{E + p^2} \\
0 \\
\sqrt{E - p^2} \\
\end{array} \right),
\]

\[
u^1 = \left( \begin{array}{c}
\sqrt{E - p^2} \\
0 \\
-\sqrt{E + p^2} \\
0 \\
\end{array} \right), \quad v^2 = \left( \begin{array}{c}
0 \\
\sqrt{E + p^2} \\
0 \\
-\sqrt{E - p^2} \\
\end{array} \right). \quad (6.72)
\]

These are the solutions for the spinor part only. The complete solution includes the spinor and spacetime parts, where the spinors of (6.72) are multiplied by the appropriate exponential function as in (6.66).

The inner product of two spinors can be calculated from (6.71):

\[
\langle \xi | \xi \rangle = 2 E \xi^\dagger \xi.
\]

But this is not Lorentz invariant, as $E$ depends on the observer. So instead we define the \textit{adjoint} function\(^{11}\)

\[
\bar{u}(p) = u^\dagger(p) \gamma^0. \quad (6.73)
\]

One usually uses a basis $\xi^1 = \left( \begin{array}{c}
1 \\
0 \\
\end{array} \right)$ and $\xi^2 = \left( \begin{array}{c}
0 \\
1 \\
\end{array} \right)$, so that $\xi^s \xi^r = \delta^{sr}$. Then

\[
\bar{u}^s(p) u^r(p) = 2m \xi^s \xi^r = 2m \delta^{sr}, \quad (6.74)
\]

which is the standard normalization for the spinor inner product of Dirac spinors.\(^{91}\)

Now an arbitrary particle solution of the Dirac equation can be represented by a mode expansion composed of an integral over momentum and a sum over spin states, including particles (a terms) and antiparticles (b terms).\(^{92}\)

The subscripts on the coefficient operators indicate the corresponding momentum, $a^\dagger_p | \psi \rangle = |p \rangle$. Thus

---

\(^90\) Schwartz, 188-189.

\(^91\) Schwartz, 191.

\(^92\) Seems to be mixture of Blundell and Lancaster, 102, 133 and Klauber, 103. [?]
\[ \psi(x) = \int \frac{d^3p}{(2\pi)^\frac{3}{2}} \frac{1}{(2E_p)^\frac{1}{2}} \sum_{s=1}^{2} (a_{sp}u_s(p)e^{-ip\cdot x} + b^\dagger_{sp}v_s(p)e^{ip\cdot x}) \]

(6.75a)

and

\[ \bar{\psi}(x) = \int \frac{d^3p}{(2\pi)^\frac{3}{2}} \frac{1}{(2E_p)^\frac{1}{2}} \sum_{s=1}^{2} (a^\dagger_{sp}\bar{u}_s(p)e^{ip\cdot x} + b_{sp}\bar{v}_s(p)e^{-ip\cdot x}). \]  

(6.75b)

The \( p \) subscript indicates momentum. The sum over \( s \) takes into account the two dimension of the spinors. When we do canonical quantization and consider these fields as operators, then each \( \psi \) and each coefficient should get a hat.

6.8.4. The Dirac equation in QFT

There is no classical Lagrangian or Hamiltonian density for spin-\( \frac{1}{2} \) particles. Symmetry and compatibility (with the scalar case) considerations have led to the equation for the Dirac Lagrangian of (6.51):

\[ \mathcal{L} = \psi(i\gamma^\mu\partial_\mu - m)\psi. \]

That this Lagrangian density be valid can be shown by substituting it into Euler-Lagrange equations

\[ \frac{\partial}{\partial x^\mu} \left( \frac{\partial\mathcal{L}}{\partial \dot{\phi}^\mu} \right) - \frac{\partial\mathcal{L}}{\partial \phi} = 0 \quad \text{with} \quad \phi^1 = \bar{\psi}, \phi^2 = \psi \]

(6.76)

The conjugate momenta to \( \psi \) and \( \bar{\psi} \) are then

\[ \pi = \frac{\partial\mathcal{L}}{\partial \dot{\psi}} = i\bar{\psi}\gamma^0 = i\psi^\dagger\gamma^0\gamma^0 = i\psi^\dagger, \quad \bar{\pi} = \frac{\partial\mathcal{L}}{\partial \dot{\bar{\psi}}} = 0. \]

(6.77)

So the \textbf{Dirac Hamiltonian density} turns out to be\(^9\)

\[ \mathcal{H} = \pi \bar{\psi} + \bar{\pi} \psi - \mathcal{L} = -i\bar{\psi}\gamma^0\partial_t\psi + m\bar{\psi}\psi. \]

(6.78)

Since there are no macroscopic spinor fields, there are no Poisson brackets for them. So the next step of quantization must take place by analogy. It turns out that commutation relations like those of (6.51) don’t work for spin-\( \frac{1}{2} \) particles, but the corresponding \textbf{anti-commutation} relations do. So with \( \{a, b\} = ab + ba \), the anti-commutation relations for spin-\( \frac{1}{2} \) fields are

\[ \{\hat{\psi}_a(t, \vec{x}), \hat{\psi}_b^\dagger(t, \vec{y})\} = \delta^3(\vec{x} - \vec{y})\delta_{ab}, \]

\[ \{\hat{\psi}_a(t, \vec{x}), \hat{\psi}_b(t, \vec{y})\} = \{\hat{\psi}_a^\dagger(t, \vec{x}), \hat{\psi}_b^\dagger(t, \vec{y})\} = 0. \]

(6.79)

The same relations with \( \bar{\psi} \) instead of \( \psi^\dagger \) require a factor of \( \gamma^0 \) on the right. The hats on the fields indicate that they are considered to be operators. Application of these relations to the spinors of (6.75) shows that the coefficients must also obey the commutation relationship

\[ \{\hat{a}_s(p), \hat{a}^\dagger_r(q)\} = \{\hat{b}_s(p), \hat{b}^\dagger_r(q)\} = \delta^3(p - q)\delta_{sr} \]

and so are creation and annihilation operators, as we found for scalar (Klein-Gordon) particles.

The Hamiltonian density can be integrated over all of space and three pages of math\(^4\) to give the free Dirac Hamiltonian.

---

93 Klauber, 104.
94 Klauber, 105-108.
which is the sum of the energies (considered positive) of the particles and antiparticles.\textsuperscript{95}

As for the Klein-Gordon coefficients (Table 3), \( \hat{a}^\dagger \) and \( \hat{b}^\dagger \) create a single particle and antiparticle, respectively; \( \hat{a} \) and \( \hat{b} \) destroy a particle or antiparticle, respectively. Then the \( \psi \) of (6.75) destroys particles and creates antiparticles, so it is the \textit{total particle-annihilation operator}; whereas \( \psi^\dagger \) creates particles and destroys antiparticles and so is the \textit{total particle-creation operator}.

However, (6.80) shows that particles of both types a and b have positive energies (aside from the vacuum energy), unlike the case of RQM where d particles have negative energy. But we are left with the same infinite-energy problem for the vacuum expectation value (VEV) of the energy we had for the K-G equation.

That is why no more than one spin-\( \frac{1}{2} \) particle can occupy the same state. It is due to the anti-commutation rules for such states. Adding a particle to a state which already has one would be\textsuperscript{96}

\[ a_i^\dagger(p) \langle \psi(p) \rangle = a_i^\dagger a_i^\dagger |0\rangle. \]

But since

\[ \{a_i^\dagger, a_i^\dagger\} = 0, \]

trying to create a second particle in the same state as the first destroys the state. This is the \textbf{Pauli exclusion principle}. It says that fermions make up the stuff around us. And it comes from the anticommutation relations for spinors.

### 6.8.5. Feynman propagator

Similarly to what was done with bosons, we can (rather laboriously) calculate the Feynman propagator for spin-\( \frac{1}{2} \) particles:

\[ S_F(x - y) = \frac{1}{(2\pi)^4} \int_{-\infty}^{+\infty} e^{-ip(x-y)(\hat{p} + m)} \frac{d^4 p}{p^2 - m^2 + i\epsilon}, \]  

(6.82)

which in the 4-momentum form is

\[ S_F(p) = \frac{(\hat{p} + m)}{p^2 - m^2 + i\epsilon} = (\hat{p} + m)\Delta_F(p). \]  

(6.83)

### 6.9. Spin-1 particles – photons

Using the same methods as for spin-0 and spin-\( \frac{1}{2} \), we can find the relevant equations for photons. We start directly with relativistic EM because there is no non-relativistic version, EM being already Lorentz-i-invariant and so the precursor of and inspiration for SR.

From (6.12), the charge-free Lagrangian for the EM field is

\[ \mathcal{L}^1_0 = (\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu). \]  

(6.84)

The simplest possible QFT Lagrangian density, as proposed by Fermi\textsuperscript{97}, is patterned after the classical,
relativistic Lagrangian.

\[ \mathcal{L}_0^1 = -\frac{1}{2}(\partial_\nu A_\mu(x)(\partial^\nu A^\mu(x)). \]  

(6.85)

Putting the Euler-Lagrange equations to work on this leads

\[ \partial_\alpha \partial^\alpha A^\mu(x) = 0, \]  

(6.86)

which is the covariant form of Maxwell’s equations in the Lorenz gauge. If \( A^\mu \) is real (it is), it has plane-wave discrete solutions\(^98\)

\[ A^\mu(x) = \sum_{r,k} \frac{1}{\sqrt{2V\omega_k}} \mathcal{E}_r(\vec{k})a_r(k)e^{-iwx} + \]

\[ \sum_{r,k} \frac{1}{\sqrt{2V\omega_k}} \mathcal{E}_r(\vec{k})a^\dagger_r(k)e^{ikx} \]  

(6.87)

\[ = A^{\mu+} + A^{\mu-} \]

and continuous ones\(^99\)

\[ A^\mu(x) = \sum_r \int \frac{d^3 \vec{k}}{\sqrt{2(2\pi)^3\omega_k}} \mathcal{E}_r(\vec{k})a_r(k)e^{-iwx} + \]

\[ \sum_r \int \frac{d^3 \vec{k}}{\sqrt{2(2\pi)^3\omega_k}} \mathcal{E}_r(\vec{k})a^\dagger_r(k)e^{ikx} \]  

(6.88)

\[ = A^{\mu+} + A^{\mu-} \]

There is no second set of solutions in \( b_r(\vec{k}) \), as for the K-G equation, because the \( A^\mu \) are real, since they are the basis of the observables \( \vec{E} \) and \( \vec{B} \), the electric and magnetic fields. This is basically because they are their own antiparticles.\(^100\) In these equations, both \( r \) and \( \mu \) take on integral values from 0 to 3. The solution \( A^\mu \) is a vector (a 4-vector), so photons are referred to as vector bosons.

The \( \mathcal{E}_r^\mu \) are the components of polarization vectors \( \mathcal{E}_r \). Being 4-vectors, each \( \mathcal{E}_r \) has 4 components \( \mathcal{E}_r^\mu \), labeled by the superscript \( \mu \). They are generally taken to be orthogonal. Let’s adopt a coordinate system, somewhat simply called the photon aligned coordinate system\(^101\), in which the 3-vector \( \mathcal{E}^r \) components lie along the x, y and z axes, with the 3\(^{rd} \) component along the z axis which we also take to be the direction of \( \vec{k} \).

Since \( A^\mu \) is real, we consider only the \( r=1 \) (\( \mathcal{E} = 0, 1, 0, 0 \)) polarization, so the result of equation (6.87) is\(^102\)

\[ A^\mu(x) = \sqrt{\frac{2}{V\omega_k}} \mathcal{E}_1(\vec{k})a_1(k)\cos{x^3} \]

where \( x^3 = z \) so \( A^\mu \) is a cosine wave in the x-z plane, an example of linear polarization. The magnetic field \( \vec{B} = \nabla \times \vec{A} \) is then perpendicular to it in the y-z plane.\(^103\)

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98 Klauber, 147.
99 Blundell and Lancaster, 122.
100 Klauber, 141-2, 148.
101 Klauber points out it should be called the “photon-polarization vector-axes aligned” system.
102 Klauber, 143.
103 For a complete discussion, with diagrams, see Klauber,142-143.
Care must be taken in order to avoid confusing spin and polarization. QFT for photons is formulated in terms of polarization; for fermions, of spin. Nevertheless, it is as if each type of particle has what might be called a “pseudo-angular momentum” or spin factor which multiplies its state vector components. For scalars (K-G) this is of dimension 0, for spinors 1/2 and for photons 1, corresponding to the “spin” in all three cases.

As pointed out in section 6.8.1, SR requires that the spin of a photon, because it is massless, be aligned along its direction of motion, either forwards or backwards, and so lies along $+\hat{k}$ or $-\hat{k}$. Polarization vectors, however, have four possible mutually orthogonal states in 4-d space.

The equations (6.87) and (6.88) resemble those for the K-G equation, except that:

- photons have no mass ($\mu = 0$ in K-G equation);
- the electric and magnetic fields derived from the 4-vector potential $A^\mu$ are real and therefore $A^\mu$ is too;
- $A^\mu$ is a 4-vector.

In fact, in QM photons are their own anti-particles and this is the meaning of the reality of $A^\mu$.

Second quantization leads unsurprisingly to commutation relations for the coefficients of the photon\textsuperscript{104}

\begin{equation}
[a_r(\vec{k}), a_s^\dagger(\vec{k}')] = \zeta_r \delta_{\vec{k},\vec{k}'} \delta_{r,r'} \quad \text{with} \quad \zeta = (-1, 1, 1, 1) \quad \text{(discrete)} \quad (6.89)
\end{equation}

\begin{equation}
\zeta_r \delta_{\vec{k},\vec{k}'} \delta(\vec{k} - \vec{k}') \quad \text{(continuous)} \quad (6.90)
\end{equation}

where underlined subscripts are not summed. The factor $\zeta_r$ comes from the fact that

\[-\zeta_r \delta_{\vec{k},\vec{k}'} = \eta_{rs},\]

the Minkowski metric, which arises naturally from EM’s already being Lorentz-invariant.

The analogy with the other two particle types does not stop here. Lo, behold, the Hamiltonian is

\begin{equation}
H = \sum_{\vec{k},r} \omega^s_{\vec{k}} \left(N_r(\vec{k}) + \frac{1}{2}\right), \quad (6.91)
\end{equation}

where the \textbf{number operator} is

\begin{equation}
N_r(\vec{k}) = \zeta_r a_r^\dagger(\vec{k}) a_r(\vec{k}). \quad (6.92)
\end{equation}

As expected

- $a_r^\dagger(\vec{k})$ creates a photon with momentum $\vec{k}$ and polarization $r$, and
- $a_r(\vec{k})$ destroys a photon with momentum $\vec{k}$ and polarization $r$.

And so forth. Last but not least, the photon propagator is expressed as follows:

\begin{equation}
D_F^{\mu\nu}(x - y) = \frac{-g^{\mu\nu}}{(2\pi)^4} \int \frac{e^{-ik(x-y)}}{k^2 + i\epsilon} \, d^4 k, \quad (6.93)
\end{equation}

in physical space, and

\begin{equation}
D_F^{\mu\nu}(x - y) = \frac{-g^{\mu\nu}}{k^2 + i\epsilon}, \quad (6.94)
\end{equation}

in momentum space.

\textsuperscript{104} Klauber, 148.
7. Symmetries and the Standard Model

Now that we've studied the equations and their solutions for free scalars, spinors and vector particles, let's get back to symmetries and their effects. The reason for all this is to understand particle physics. That is expressed nowadays by what is called the Standard Model, which in turn is part of the Core Theory. The SM is explained by QFT which depends on QM. OK?

7.1. The standard model (SM)

Particles are either fermions or bosons, according to the type of statistics they obey. ("Type" and "family" are not official terms of the SM.)

**Bosons** have spin which takes on only integral values – for instance, 0 or 1. They obey Bose-Einstein statistics.

**Fermions** all have half-integral values of spin – $\frac{1}{2}$, $\frac{3}{2}$ and so on. They obey Fermi-Dirac statistics. Most importantly, they are the basic components of matter, the blocks from which all of us are built. This is due to their being constrained by the Exclusion Principle not to occupy the same state, as we saw by 6.81.

There are two types of basic fermions – **quarks** and **leptons**. They all are shown in Figure 2, the particle zoo, which is composed of six quarks (shown in purple), six leptons (green) and four gauge bosons (orange). Table 4 displays some of their properties more visibly.

---

6.81 Figure 2: Elements of the standard model

Physics recognizes four forces, three of which are considered by the Standard Model.

The strong force is mediated by quarks, so hadrons, particles composed of quarks, are subject to it. Hadrons are divided into mesons ("middle weight") and baryons ("heavy", meaning mass \( > \) the proton mass). The difference between mesons and baryons is not just an imprecise question of mass, but of the number and type of quarks in them. Mesons are composed of two quarks, one quark and the corresponding antiquark; baryons, of three quarks.

Fermions are particles of half-integer spin and include the hadrons, but also the "lighter" leptons. Leptons are not composed of quarks and so do not interact via the strong force, although the charged ones are subject to the EM force.

Quarks are distinguished by their flavor – up, down, charm, strange, top or bottom – which obviously are arbitrary terms and have nothing to do with taste or appearance. They are arranged in three columns called generations. Quarks have charges +2/3 or -1/3; leptons, 0 or ±1. Each of the six flavors of quark exists in three versions indicated (by analogy) by the colors red, green and blue, for a total of 18 combinations. When forming matter particles, the quarks must group together in such a way that the result is "colorless", so they occur in the combination R+G+B for baryons like the neutron or proton, and C+C, where the bar above the character indicates an antiparticle, for bosons. Color is important because it allows the existence within a nucleon of three

---

### Table 4. Particles of the Standard Model

| Family     | Type      | Generation | Spin (|) | Particles (charge or quarks)                             |
|------------|-----------|------------|-------|----------------------------------------------------------|
| fermions   | leptons   | 1          | \( \frac{1}{2} \) | \( e^- (1), \nu_e (0) \)                               |
|            |           | 2          | \( \frac{1}{2} \) | \( \mu^- (1), \nu_\mu (0) \)                        |
|            |           | 3          | \( \frac{1}{2} \) | \( \tau^- (-1), \nu_\tau (0) \)                     |
| quarks     |           | 1          | \( \frac{1}{2} \) | \( u^+ (\frac{2}{3}), d^- (-\frac{1}{3}) \)     |
|            |           | 2          | \( \frac{1}{2} \) | \( c^+ (\frac{2}{3}), s^- (-\frac{1}{3}) \)     |
|            |           | 3          | \( \frac{1}{2} \) | \( t^+ (\frac{2}{3}), b^- (-\frac{1}{3}) \)     |
| composite  |           | \( \frac{1}{2} \) | \( \frac{3}{2} \) | \( \Delta^{++} (uuu), \Delta^- (ddd) \) |
| bosons     | Higgs     | 0          | \( 0 \) | Higgs                                                    |
| gauge bosons |          | 1          | \( 1 \) | \( \gamma \) (photon), \( Z^0, W^\pm, g_1 \) (gluons) |
|            | graviton  | 2          | \( 0 \) | \( \Delta^- (ddd) \) |
| composite (mesons) |          | 0          | \( 0 \) | \( \pi^+ (u\bar{d}) \) |
|            |           | 1          | \( 1 \) | \( K^- (\bar{u}s) \) |
|            |           | 1          | \( 1 \) | \( \rho^+ (u\bar{d}) \) |
|            |           | 1          | \( 1 \) | \( K^{*-} (\bar{u}s) \) |

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106 So called by Murray Gell-Mann, the same guy who found the word “quark” while reading *Finnegans Wake*. Uh-huh...
107 Not the same as the colors on the chart.
quarks which would otherwise be in the same state and so would be forbidden by the Exclusion Principle. Antiquarks have anticolors.

The mass of, for instance, a proton (about 938 MeV/c\(^2\)) is far greater than the sum of the masses of its constituent quarks (3x2.3 MeV/c\(^2\) \(\approx\) 7 MeV/c\(^2\)). The difference, almost 99% of the proton’s mass, is potential energy of the strong force which binds the quarks together into the proton. Splitting the proton into quarks requires this amount of energy, which explains why physicists need such powerful particle accelerators.

Matter is made up of atoms with nuclei containing protons and neutrons (together called \textit{nucleons}), with electrons forming a negatively-charged cloud around the nucleus. Such long-lived particles are made up of quarks of the first generation. A proton is composed of two up quarks and a down quark. The former have a charge of +2/3, the latter of -1/3, so the total is +1. A neutron is composed of an up and two downs, for a total charge of +2/3-1/3-1/3 = 0. And so on.

Obviously, there is some overlap in these categories, as is sketched in the following figure. Although fermions and bosons are distinct, hadrons may be either fermions or bosons. Which means that some fermions and some bosons are composed of quarks. Those bosons which are not are force-carrying particles. But the particles which constitute matter are always fermions.

![Figure 3. Bosons, hadrons and fermions, from Hugo Spinelli via Wikimedia Commons](https://commons.wikimedia.org/wiki/File:Bosons-Hadrons-Fermions-RGB.svg)

To summarize, modern physics recognizes four fundamental forces, or interactions, in nature.

1. Gravity is theorized to be conveyed by a yet-to-be observed particle called the \textit{graviton} (not shown in the figure because never observed and because gravity is not part of the standard model). It is a weak force (the weakest) but works across enormous, interstellar distances and is responsible for no less than the formation of stars, galaxies and planets. Gravity is due to the curvature of space, so there is debate whether it is a force at all. In fact, we should speak not of four fundamental forces, but of four fundamental interactions.\(^\text{109}\)

2. The electromagnetic force between charges or magnets is conveyed by the \textit{photon}, which is the particle of light. Like gravity, it has infinite range, but is stronger. Since its sources can be either positive or negative charges, the two cancel each other out, making the effective force at large distances weaker than gravity. The quantum theory of the electromagnetic force is \textit{quantum electrodynamics}, or \textit{QED}. We will confront the basic equation for QED in (7.12).

3. The strong force is the strongest, but is very short-ranged. It holds quarks together in the nucleus in spite of the repulsive electric forces between proton charges. It is conveyed by the appropriately-named \textit{gluon}. The quantum theory of strong interactions is called \textit{quantum chromodynamics}, or \textit{QCD}, with which we will deal in Section 7.9.

4. The weak force is weaker than the strong or EM forces, but is still stronger than gravity. It is conveyed by the W and Z bosons. It is a very short-range force, responsible for decays of various radioactive particles. Such decay is largely responsible for the existence of the Periodic Table and so for the elements of which our Earth and we are made. Since it enables the transmutation of a proton into a neutron and leads thence to the formation of deuterium, it is essential for the “burning” which takes place during the life of a star.


in the Sun! We will study this force in Sections 7.10 and 7.11.

Weak interactions are more complex than the others. They are the only ones which exhibit parity violation. This occurs because the weak force is asymmetric for left-handed and right-handed particles. There are two kinds of weak interactions: charged interactions, mediated by $W^\pm$ bosons, and neutral ones, mediated by $Z^0$ bosons. Only the charged weak interactions can change flavor.\footnote{Griffiths, 74.}

So there are two infinite-range forces, gravity and EM, and two short-range ones, the strong and weak. By order of strength, from strongest to weakest, they are strong, EM, weak and gravity.

### 7.2. The Core Theory

It is important to understand that quantum theory alone does not explain the world. It is a framework for expressing theories about the world and for doing calculations. It can, for instance, be used to explain atoms and the periodic table, but only by adding information, e.g., that a hydrogen atom is composed of a proton with one electron moving about near it under the sway of the electromagnetic force.

Modern physics considers everything to be made up of fields. This is the idea behind quantum field theory (QFT). Fields are where the buck stops. Like the bottom turtle, they are not made up of anything else (well, as far as we know or imagine). The particles which we see as the constituents of all the stuff around us are vibrations in quantum fields, fermion fields for matter, boson fields for forces. Although the separation of GR and QM keeps us from understanding completely the realm of the infinitely big or the vanishingly tiny, the Big Bang or black holes, most of the time who cares? The stuff around us in our everyday world is just ordinary matter and not composed of black holes, so we can effectively describe the world we live in with the standard model, based on QFT and the four forces, plus GR. This is what some physicists call the Core theory.

\[
\text{Core Theory} = \text{QFT} + \text{GR}
\]

And it works, providing the physical laws underlying chemistry, biology, astrophysics, engineering and much of cosmology (i.e., except for the Big Bang and black holes).

### 7.3. Noether's theorem and currents

Consider a Lagrangian density $\mathcal{L}(\phi_j, \partial_\mu \phi_j)$ whose field variables $\phi_i$ undergo infinitesimal global transformations $\delta \phi_i$ which are functions of a parameter $\alpha$. Use of the Euler-Lagrange equations and integration by parts shows that

\[
\delta \mathcal{L} = \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_i)} \right) \delta \phi_i.
\]

If the Lagrangian is invariant under the transformation, then this quantity disappears and we have

\[
\partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_i)} \right) \delta \phi_i = 0,
\]

which is a continuity equation for a quantity

\[
J_\mu = \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_i)} \right) \delta \phi_i.
\]

In fact, it is the action, the integral of the Lagrangian, which is extremized, so we can also add to the Lagrangian a surface term, a divergence of some function $\mathcal{W}$, which will disappear when integrated over a surface sufficiently far away that fields are all zero. Then the quantity in question is the Noether current defined by

\[
\text{Noether current}
\]
Equation (7.1) shows that the current has zero four-divergence \( \partial \mu J^\mu = 0 \), so its zeroth component, \( J^0 \), integrated over all of space is conserved. One can see this as a charge,

\[
Q = \int_{\text{all space}} J^0 d^3x,
\]

which is constant in time. i.e., it is conserved. This is Noether’s theorem.

Note that in case of several fields \( \phi_i \), the RHS of equation (7.2) is to be summed over \( i \). The derivation of this current does not depend on whether \( \alpha \) is a constant, so Noether’s theorem is equally valid for global and local transformations.

Applying this with various transformations leads to beloved conservation laws, as summarized in Table 5.

<table>
<thead>
<tr>
<th>Transformation</th>
<th>Current ( J^\mu )</th>
<th>Conserved quantity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spatial translation ( q_i \rightarrow q_i + \delta q_l )</td>
<td>( \frac{\partial \mathcal{L}}{\partial q} \delta q = p \delta q )</td>
<td>Momentum ( p )</td>
</tr>
<tr>
<td>Spatial rotation ( q_i \rightarrow q_i + \epsilon_{ijk} q_j a_k )</td>
<td>( (\vec{p} \times \vec{q}) \cdot \vec{a} = \vec{L} \cdot \vec{a} )</td>
<td>Angular momentum ( \vec{L} )</td>
</tr>
<tr>
<td>Time translation ( t \rightarrow t' = t + \epsilon )</td>
<td>( p q - \mathcal{L} = \mathcal{H} )</td>
<td>Energy ( \mathcal{H} ) (Hamiltonian)</td>
</tr>
<tr>
<td>Boost ( q_i \rightarrow q_i + v t ), so ( q_i \rightarrow q_i + v )</td>
<td>( pt - m q )</td>
<td>Uh...</td>
</tr>
</tbody>
</table>

Table 5. Conserved quantities of spacetime transformations

The last boost current in the table depends on \( t \), which can be picked so that the current is zero, which is therefore conserved. Uh... okay.\(^{111}\)

Let’s pause to note that in the case of conserved momentum, for instance, a measurement of momentum is based on the generator of translations and so involves a translation, thus changing the value of the position. This realization offers a qualitative way to understand the Uncertainty Principle stating, in this case, that momentum and position cannot be simultaneously measured precisely.

Spacetime transformations of fields are messier because changes come from the fields themselves as well as from the transformed coordinates. The results, though, are consistent with the contents of Table 5. In particular, spacetime translations lead to field transformations\(^{112}\)

\[
\phi(x^\mu) \rightarrow \phi(x^\mu + a^\mu) = \phi(x) + a^\mu \partial_\mu \phi(x)
\]

and since it is a scalar, the Lagrangian must transform in similar fashion:

\[
\mathcal{L} \rightarrow \mathcal{L} + a^\mu \partial_\mu \mathcal{L} = \mathcal{L} + a^\nu \partial_\nu (\delta^\mu_\nu \mathcal{L}).
\]

Putting this together with equation (7.2) leads to a tensor current.
The time-like “charge” which is conserved is

\[ P^0 = \int T^{00} d^3x = \int d^3x \left( \frac{\partial \mathcal{L}}{\partial (\partial_0 \phi)} \partial_0 \phi - \mathcal{L} \right) = \int d^3x \left( \Pi^0 \phi - \mathcal{L} \right) \]

which is therefore interpreted as the momentum of the field in the \( I \) direction, which is also conserved. The spatial “charge” is

\[ P^i = \int T^{0i} d^3x = \int \pi \partial_i \phi \, d^3x \]

This results in the conservation of energy and each of the three components of momentum.

Applying Noether’s theorem to the equation of QED shows the current concerned is

\[ J^\mu = -g \bar{\Psi} \gamma^\mu \Psi. \]  

We will find the same result for the QED interaction Lagrangian in (7.14).

Table 6 summarizes similar results for fields if equation (7.2) is used to calculate the Noether current due to a \( U(1) \) transformation on scalar and Dirac fields.

<table>
<thead>
<tr>
<th>Type of field</th>
<th>Current</th>
<th>Conserved quantity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Free scalar (Klein-Gordon)</td>
<td>( j^\mu = i((\partial_\mu \phi)\phi^\dagger - (\partial_\mu \phi^\dagger)\phi) )</td>
<td>Charge</td>
</tr>
<tr>
<td>Free spinor (Dirac)</td>
<td>( j^\mu = \bar{\Psi} \gamma^\mu \Psi )</td>
<td>Charge</td>
</tr>
<tr>
<td>Free photon (Proca, m=0)</td>
<td>( j^\mu = 0 )</td>
<td>Charge = 0</td>
</tr>
<tr>
<td>Full QED Lagrangian (7.12)</td>
<td>( j^\mu = \bar{\Psi} \gamma^\mu \Psi )</td>
<td>Charge</td>
</tr>
</tbody>
</table>

Table 6. Conserved quantities of Noether currents for internal symmetries

The results found here using Noether’s theorem are the same as those found by the standard QM method starting with the Schrödinger equation. The result for the QED interaction equation is as expected, since photons are chargeless. The proof for the full QED Lagrangian does not depend on the spatial dependence of \( \alpha \) in the exponent of the transformation and so is equally true for local and global transformations.

Equation (7.2), when applied to changes only in the fields, not the coordinates, represents internal symmetries of the system.\(^\text{114}\) In particular, for a translation of the field,

\[ \Phi_i \rightarrow \Phi_i + \delta \Phi_i, \]

not of spacetime, it shows conservation of a new quantity, the conjugate momentum density,

\[ \pi^\nu = \frac{\partial \mathcal{L}}{\partial (\partial_\nu \Phi)}. \]

This must be distinguished from the physical momentum density of the field, which is due to invariance under

\(^{113}\) Schwichtenber, PS, 110; Blundell and Lancaster, 94-95; Peskin and Schroder, 18-19.

\(^{114}\) Klauber, 173.
spatial translations.

7.4. Isospin

It was originally proposed that the strong interaction was symmetric under exchange of a proton for a neutron or the opposite. Currently, the concept is used more for the exchange of an up quark with a down one. The idea behind it is of a new symmetry like spin and so called isospin.\textsuperscript{115} Like spin, isospin is a representation of the symmetry group SU(2), and may be expressed in representations of different dimensions, as shown in Table 1. The mathematical apparatus is the same as that of angular momentum.

Since the symmetry group is SU(2), the transformations behave like rotations from one particle identity into another. The symmetry implies the existence of a Noether current and a conserved Noether charge, the isospin. So to the extent that the strong force is invariant with respect to rotations in isospin space, isospin is conserved. Although the isospin behaves like a spin or angular momentum, it is in its own internal space and has nothing to do with these other angular-momentum-like properties. Assuming, as usual, a rotation about the 3-axis, we then can talk about a set of \( n \) particles with isospin \( I = n \) and \( I_3 = -n \ldots + n \) in integral steps.

Some examples are shown in Table 7.

<table>
<thead>
<tr>
<th>particles</th>
<th>Isospin ( I )</th>
<th>( I_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p, n ) or ( (K^0, K^-) )</td>
<td>( \frac{1}{2} )</td>
<td>( n ) or ( K^0 = -\frac{1}{2}, p ) or ( K^+ = +\frac{1}{2} )</td>
</tr>
<tr>
<td>( \nu_e, \bar{c}_L )</td>
<td>( \frac{1}{2} )</td>
<td>( \nu_e = +\frac{1}{2}, \bar{c} = -\frac{1}{2} )</td>
</tr>
<tr>
<td>( e_R )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( u, d ) (quarks)</td>
<td>( \frac{1}{2} )</td>
<td>( u = \frac{1}{2}, d = -\frac{1}{2} )</td>
</tr>
<tr>
<td>all other quarks</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \Lambda ) or ( \bar{\Omega} )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( W^-, Z_0, W^+ )</td>
<td>1</td>
<td>( W^- = -1, Z_0 = 0, W^+ = +1 )</td>
</tr>
<tr>
<td>( \pi ) mesons</td>
<td>1</td>
<td>( \pi^+ = +1, \pi^0 = 0, \pi^- = -1 )</td>
</tr>
<tr>
<td>( \Delta ) baryons</td>
<td>2</td>
<td>( \Delta^{++} = +2, \Delta^{+} = +1, \Delta^{0} = 0, \Delta^{-} = -1, \Delta^{--} = -2 )</td>
</tr>
</tbody>
</table>

Table 7. Some isospin examples.

\( I_3 \) of hadrons is easily found by summing the respective values for the constituent quarks.\textsuperscript{116} For combinations of hadrons, Clebsch-Gordon coefficients are necessary. The EM charge is given by the Gell-mann Nishijima relation of (7.49). The hypercharge is determined by the number of quarks in a hadron. In terms of the numbers of up, down, strange, charmed, bottom and top quarks:\textsuperscript{117}

\[
Y = \frac{2}{3} n_u + \frac{1}{3} n_d - \frac{2}{3} n_s + \frac{4}{3} n_c - \frac{2}{3} n_b + \frac{4}{3} n_t.
\]

The use of Feynman diagrams to describe hadron (particles composed of quarks held together by the strong force) interaction renders the calculation of this quantum number unnecessary.

\textsuperscript{115} The term isospin comes from isotopic spin. Nuclear physicists call it isobaric spin. Griffiths, 129.
\textsuperscript{116} “Isospin”. www.asc.ohio-state.edu/gan.1/teaching/winter10/Chapter5.pdf
7.5. Gauge symmetry

Global symmetries are physical and have physical effects, since they conserve charge (Noether). Gauge invariance is not physical, being “… merely a redundancy of description we introduce to be able to describe the theory with a local Lagrangian.” Gauge symmetry (or gauge invariance) allows changes in the configuration of the underlying, unobservable field(s) that have no distinguishable effects on observable properties.

The standard example is the electromagnetic vector potential \( A \), changes in which leave the electric and magnetic fields \( \mathbf{E} \) and \( \mathbf{B} \) unchanged. The unobservable field is called the gauge field and a gauge transformation changes it from one configuration, or gauge, to another. Examples are the Lorenz or Coulomb gauges for EM.

Gauge symmetries are internal symmetries, functioning in "hidden", unobservable space.

For instance, if the field in our Lagrangian changes by

\[ \phi \rightarrow \phi^{-i\alpha}, \]

the Lagrangians themselves remain unchanged (invariant). So phase changes are changes of basis which do not affect the observable variables. Such a transformation is a global one, because \( \alpha \) is a constant and so has the same value everywhere. Gauge symmetries, though, can be transformations which are functions of location (if \( \alpha \) not constant) and so are local symmetries. The transformation indicates how the transformed function changes with respect to position (Imagine a rotation through an angle which depends on the position of the object being rotated.) and so connects together different locations. This is then a local transformation and is said to define connections between points, usually expressed in terms of a covariant derivative. As in the case of GR, connections can be thought of as measuring the curvature of space caused or measured by the gauge field which is the connection. So it can be found by parallel transporting the field around a small, closed loop.

7.6. Interaction Lagrangians – U(1) symmetry and QED

Symmetry considerations are not only valid for finding the form of free-particle Lagrangians, they can also be used to find interaction terms. It is obvious that a unitary transformation

\[ \Psi \rightarrow \Psi' = e^{i\alpha \Psi} \]

does not change the Dirac Lagrangian. Since this transformation does not affect spacetime, it represents an internal symmetry of the system, say an electron. It is the same everywhere in spacetime and so is a global transformation. But SR and its speed limit forbid such a global transformation, so let’s try a local transformation by letting \( \alpha \) depend on the coordinates. Then

\[ \Psi \rightarrow \Psi' = e^{i\alpha(x)} \Psi \quad \text{and} \quad \bar{\Psi} \rightarrow \bar{\Psi}' = \bar{\Psi} e^{-i\alpha(x)}. \quad (7.7) \]

Such conversion of a global symmetry to a local one is sometimes referred to as gauging the symmetry. Now the derivative in the Lagrangian causes an extra term under the transformation and

\[ L'_{Dirac} = L_{Dirac} - (\partial^\mu a(x)) \gamma_\mu \Psi, \quad (7.8) \]

so the Dirac spin-\( \frac{1}{2} \) Lagrangian by itself is not invariant under local U(1) transformations. The extra term should remind us of the Dirac 4-current from QFT from (6.82), \( \psi \gamma^\mu \psi \), multiplied by a partial derivative.

The Proca Lagrangian for a massless spin-1 (vector) particle has a local U(1) symmetry when

\[ 118 \text{ Schwarz, 130-131.} \\
119 \text{ Klauber, 136.} \\
120 \text{ Schwichtenberg shows an example of this. NNOFT, 129-35.} \\
121 \text{ Be careful. Some texts use } \Psi \rightarrow \Psi' = e^{-i\alpha(x)} \Psi, \text{ which changes the sign of the interaction term.} \]
This is of course the usual gauge symmetry of the electromagnetic field.

Now comes the less obvious step. Knowing that a photon has spin 1 suggests that we introduce the vector field into the Dirac Lagrangian by defining a covariant derivative

\[ D_\mu = \partial_\mu + iqA_\mu. \tag{7.10} \]

When we replace the partial derivative by the covariant derivative (7.10) and use the transformation (7.9) of the vector field, the Dirac Lagrangian is indeed invariant under a local U(1) transformation.\(^\text{122}\)

Note that with requirements (7.9) and (7.10) satisfied,

\[(D_\mu \Psi)' = D'_\mu \Psi' = U(x)D_\mu U^{-1}(x)U(x)\Psi = U(x)D_\mu \Psi(x).\]

This is why \(D_\mu\) is called the covariant derivative, as the form of the equation is not changed by the transformation.\(^\text{123}\)

A word on the covariant derivative. We use the derivative of a function to pass from its value at one point in spacetime to a point at an infinitesimal distance from there. If a field is submitted to a transformation which varies locally, something is needed to keep track of that change from point to point. Such objects are called connections. They should not be confused with the Christoffel connections of GR, although those have a logically similar function, serving to keep track of the changes of the basis vectors in terms of the metric.\(^\text{124}\)

Equations (7.7) and (7.9) express the local changes in and in . The connection may be expressed as an additional term to the derivative, which then becomes the covariant derivative of (7.10). The connection introduces the vector field and in a sense is the vector field, give or take a proportionality constant. The vector field, in this case, is the photon. The connection necessary to take into account the local gauge transformation of the Dirac Lagrangian is the photon.

Making a transformation local means the equations of physics must take into account the change in the transformation itself. This requires the introduction of a new field, a gauge field, in the derivative, which becomes the covariant derivative (7.10). The change in the transformation itself is expressed in the transformation of the gauge field itself, as in (7.9). Including the gauge field and taking into account its transformation makes the Lagrangian locally symmetric (invariant).

Equation (7.10) should be called a gauge covariant derivative, to distinguish it from the GR covariant derivative, but this appellation is rarely used. On the one hand, this is just a notational device to make the equation simpler: It avoids our writing the interaction term specifically. On the other, it tells us we need to know more than just the behavior of the free Lagrangian.

Meanwhile, at U(1), we have a Lagrangian which is locally invariant under U(1) transformations:

\[ \mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi + qA_\mu \bar{\psi}\gamma^\mu \psi. \]

There is a problem, though. Because there are no terms in the derivative of \(A_\mu\), the use of (7.2) to calculate the Noether current for the vector field results in zero, which does not help us to determine the dynamics of the vector field. So we have to add in the field itself and the way to do this is to add its own massless Proca term to the Lagrangian. The final result for this interaction is:

\[ \mathcal{L}_{\text{Dirac+Proca+int}} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi + qA_\mu \bar{\psi}\gamma^\mu \psi - \frac{1}{2}(\partial^\mu A^\nu \partial_\mu A_\nu - \partial^\mu A^\nu \partial_\nu A_\mu) \]

\(^\text{122}\) Robinson does this succinctly but clearly on 202-5.

\(^\text{123}\) Schwichtenberg, PS, 146.

\(^\text{124}\) According to Sean Carroll, the connections can be calculated, as in GR, by considering the parallel transport of a vector in the space. https://www.preposterousuniverse.com/blog/2020/06/30/the-biggest-ideas-in-the-universe-15-gauge-theory/
Now the Lagrangian is invariant under local SU(1) transformation, which requires transforming both the Dirac and the Proca terms according to (7.7) and (7.9). The first term in (7.11) is the Dirac Lagrangian; the second, the gauge (vector) potential multiplied by the Dirac 4-current (6.82) and the third, the Maxwell Lagrangian for a massless particle.

Note that the interaction term $L_I$ contains elements of both the Dirac and Proca (vector) Lagrangian, which seems right for an interaction between the two particles.

This technique (or recipe) of adding together free fields and then using the covariant derivative to bring out their interaction is called \textit{minimal coupling}, since it ignores such things as magnetic moments. A theory which introduces a field, $A^\mu$, to bring about local invariance is a \textit{gauge theory}, so the field $A^\mu$ is the \textit{gauge field}. As one book puts it, the gauge field is “... designed to roll around spacetime cancelling [sic] out terms that stop the theory from being invariant.”

Now the Lagrangian is the correct \textit{Lagrangian for the quantum field theory of electrodynamics}, QED:

$$L_{\text{Dirac+Proca+int}} = \bar{\Psi}(i\gamma_\mu D^\mu - m)\Psi - \frac{1}{2}(\partial^\mu A^\nu A_\mu A_\nu - \partial^\mu A^\nu \partial_\nu A_\mu)$$

$$= \bar{\Psi}(i\gamma_\mu D^\mu - m)\Psi - \frac{1}{4}(F^{\mu\nu}F_{\mu\nu}). \quad (7.12)$$

All QED (as well as classical EM, of course) comes from Lorentz and local U(1) invariance + a bit of imagination. Symmetry under a local U(1) transformation required the introduction of a gauge field $A^\mu$.

We could write the equation using the covariant derivative for the components of the vector potential also, but the differences cancel out. Note that equation (7.11) and both variants of (7.12) represent the same physical system.

Recap:

1. Start with a Lagrangian for a spinor, which is invariant under a global U(1) transformation.
2. Gauge it, i.e., make the transformation local.
3. In order for the Lagrangian to be invariant under the local U(1) transformation, use the covariant derivative (7.10) to introduce the vector field $A_\mu$, and
4. $A_\mu$ transforms like (7.9), i.e., like a spin-1 particle under the same local U(1) transformation. Now the Lagrangian is locally invariant under SU(1).
5. Add in a Proca Lagrangian for the spin-1 particle so that the Noether charge will be non-zero.

Adding local phase invariance to the Dirac field requires the inclusion of a massless vector field. The result is all of electrodynamics. If you don’t find this amazing, you should.

The interaction term can understood by starting with Maxwell’s equations in tensor form and the Lorenz gauge (i.e., with currents) and including the electron charge:

$$\partial_\alpha \partial^\alpha A^\mu(x) = -e j^\mu(x). \quad (7.13)$$

We have already seen that the current for a Dirac electron is given by

$$j^\mu = (\rho, \vec{j}) = \bar{\psi}\gamma^\mu \psi \quad \text{with} \quad \partial_\mu j^\mu = 0, \quad (6.82)$$

so from (7.13) for a photon, we can write

125 Lancaster and Blundell, 128.
126 Robinson’s section 4.5.3 is an excellent and concise presentation of this gauging of the spinor field.
127 Following Klauber.
We have just linked Maxwell and Dirac by using the latter’s electron current density from (6.82) in the former’s continuity equation. In other words, we have assumed that the Maxwell current is composed of Dirac electrons, by which we have linked the two and showed the interaction between them. The result is that, in general, \( A_\mu \) and \( \psi \) are no longer independent of each other.

From the Lagrangian (7.12), the Euler-Lagrange equation for \( \bar{\psi} \) leads us to the interaction form of the Dirac equation.

\[
(i\gamma^\mu \delta_\mu - m)\psi = -e\gamma^\mu \psi A_\mu. \tag{7.15}
\]

This has been applied to the hydrogen atom by assuming no magnetic field and pure Coulomb charge, so that

\[
A_\mu = (\Phi, 0), \quad \Phi = \frac{e}{4\pi r}.
\]

The result of the analysis gives a relativistic fine-structure that correctly describes the observed spectrum.\(^{128}\)

In quite a similar way, one can construct the locally gauge-invariant Lagrangians for a massive charged scalar field with a massless vector field and for a massive vector field with a massless one.\(^{129}\) So locally gauge-invariant Lagrangians are available for the interaction of a massive scalar, Dirac or vector field with a massless vector field.

In fact, as Robinson puts it: “Starting with a a non-interacting Lagrangian that is invariant under the global SU(N), we can gauge the SU(N) to create a theory with a gauge field (or synonymously a ‘force carrying field’) \( A^\mu \), which is an \( N \times N \) matrix. Hence, every Lie group gives rise to a particular gauge field (which is a force carrying particle, like the photon), and therefore a particular force. For this reason, we discuss forces in terms of Lie groups, or synonymously Gauge Groups.”

We will see shortly (in section 7.11) how breaking a local symmetry leads to a gauge field’s taking on mass. That is the importance of these concepts in physics:

*Group theory and gauge transformations (or gauge theory) explain all the forces of nature.*\(^{130}\)

### 7.7. SU(n) gauge invariance

In order to ensure the local gauge invariance under U(1) symmetry for a spin-\(1/2\) particle, three things were required:\(^{131}\)

- the introduction of a massless vector (spin-1) field, which we take to represent a photon, including its free Lagrangian;
- the addition of an interaction term depending on both types of fields (through the use of the the covariant derivative); and
- the taking into account of the different transformation representations of the spinor and vector fields (the latter being the same transformations used for the vector and scalar potentials of classical electromagnetism).

This was expressed in equations (7.9), (7.10) and (7.12).

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\(^{128}\) Klauber, 185-6.

\(^{129}\) Schwichtenberg, PS, 142-3.

\(^{130}\) In principle, this includes gravity, but I’ve never seen it demonstrated.

\(^{131}\) Griffiths, Elementary particles, 360.
Equations (7.9) and (7.10) can be generalized to arbitrary dimensions. For any group SU(n), composed of \( n \times n \) matrices, with \( n^2 - 1 \) generators and of rank \( n - 1 \), a general transformation will be \( \phi_i \rightarrow U_{ij} \phi_j \) and the gauge fields transform like

\[
A_\mu \rightarrow U(x)A_\mu U^\dagger(x) + \frac{i}{g} U(x)\partial_\mu U^\dagger(x). \tag{7.16}
\]

Then if we write the transformation as

\[
U(x) = e^{-ig\Gamma^a T^a},
\]

where \( \Gamma^a \) are the \( n^2 - 1 \) real parameters and \( T^a \) are the generators, the covariant derivative will be

\[
D_\mu = \partial_\mu - igA_\mu. \tag{7.17}
\]

Remember that for \( n>1 \), the generators are matrices and may not commute, meaning we are dealing with non-Abelian groups. In this case, the field tensor is

\[
F_{\mu\nu}(x) = \frac{i}{g} [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu], \tag{7.18}
\]

where the commutator term vanishes for SU(1). From these last equations, we see that a Lie group implies the existence of a gauge field (7.16) which represents a force. This is summarized in Table 8.

<table>
<thead>
<tr>
<th>Lie group</th>
<th>No. gauge fields</th>
<th>Fundamental representation</th>
<th>Particle(s)</th>
<th>Conserved charge</th>
</tr>
</thead>
<tbody>
<tr>
<td>U(1)</td>
<td>1</td>
<td></td>
<td>massless photon</td>
<td>electric charge</td>
</tr>
<tr>
<td>SU(2)</td>
<td>3</td>
<td>( \tau = \frac{1}{2} \sigma, \sigma = \text{Pauli matrices} )</td>
<td>massive (with Higgs) W and Z bosons</td>
<td>isospin</td>
</tr>
<tr>
<td>SU(3)</td>
<td>8</td>
<td>( \frac{1}{2} \lambda, \lambda = \text{Gell-Mann matrices} )</td>
<td>massless gluons</td>
<td>color charge</td>
</tr>
</tbody>
</table>

Table 8: Summary of Lie groups and gauge fields

The representation of a group SO(n) or SU(n) consisting of \( n \times n \) matrices is called the fundamental representation.

We can go further, although the calculations are more laborious and the results, more complicated. This is because the SU(2) and SU(3) groups, being matrices and not simple exponentials, are non-Abelian. Gauge theory of such fields is Yang-Mills theory.

### 7.8. SU(2) gauge invariance

We can study local SU(2) gauge invariance for two equal-mass Dirac fields by expressing them as one two-component column vector, or doublet. A unitary matrix then can be expressed in the form

\[
U = e^{iH},
\]

---

132 After Robinson, 240-1.
133 After Schwichtenberg, 133.
134 Using notation of Blundell and Lancaster, 425-9. See also Griffiths, op. cit., 361-6

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where $\mathcal{H}$ is Hermitian. We can use the Pauli matrices, $\sigma$, to form a basis, so that

$$H = \theta \mathbf{1} + \frac{i}{2} \mathbf{\sigma} \cdot \mathbf{a}.$$

For local SU(2) symmetry, the $\alpha$s in the second term are functions of position. In this case, the derivative of the SU(2) unitary operator will bring down a factor proportional to a weighted sum of the three Pauli matrices, $\sigma$, and so the calculation requires not one, but three massless vector fields in order to guarantee invariance. The covariant derivative then can take on the form:

$$D_\mu = \partial_\mu - \frac{i}{g} \mathbf{\sigma} \cdot \mathbf{W}_\mu.$$  (7.19)

In order to make the Lagrangian invariant under local SU(2) transformations, the vector fields must transform like

$$\tau \cdot W_\mu \rightarrow \tau \cdot W_\mu + \frac{i}{g} \left( \partial_\mu \alpha \right) - \tau \cdot (\alpha \times W_\mu).$$  (7.20)

To complete the comparison with the SU(1) Lagrangian of (7.12), we need the force tensor

$$G_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu + g(W_\mu \times W_\nu).$$  (7.21)

Then the locally invariant SU(2) Lagrangian is

$$\mathcal{L} = \bar{\Psi} (i \gamma^\mu D_\mu - m) \Psi - \frac{1}{4} G_{\mu\nu} \cdot G^{\mu\nu},$$  (7.22)

Remember, this is for two equal-mass Dirac fields interacting with three zero-mass vector gauge fields. In this case, the equal-mass requirement makes the identification of the fields as known particles difficult. Also, the best candidates for the gauge fields, the W and Z bosons, have masses around 100 GeV, nowhere near zero. To get out of this dilemma, we will need the notion of symmetry breaking, coming soon in Section 7.11.

### 7.9. SU(3) symmetry and QCD

Requiring local gauge invariance for three equal-mass Dirac fields, expressed as a three-component column vector and so under U(3) symmetry, will require the addition of $3^2 - 1 = 8$ massless vector fields. Here the three Dirac particles are identified with three quarks of the same flavor (and so mass) but different colors (red, blue, green) and the vector fields to be the gluons of the strong interaction force. Each gluon forms the source for a color current, in the sense of a Noether current. 135 This set of particles forms the basis for quantum chromodynamics or QCD.

Note that both these cases require the existence of vector bosons, or gauge bosons. Gauge bosons are always vector bosons. 136 In these examples, they are massless, but one can invent (unrealistic?) massive examples also. 137

Now the state vector 138 representing three equal-mass particles is

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \quad \bar{\psi} = \begin{pmatrix} \bar{\psi}_1 \\ \bar{\psi}_2 \\ \bar{\psi}_3 \end{pmatrix}$$

so the Lagrangian looks just like the one-particle case but is in fact a three-component column vector each element of which is a four-component Dirac spinor. (Got that?) The symmetry group of this beast is U(3) with

---

135 Griffiths, 366-9. The following triplet example is inspired by these pages.
137 Schwichtenber, 150, presents the example of two massless Dirac fields (What would that be?), which requires three massive gauge bosons.
138 Or whatever it is called...
so $U^\dagger U = 1$, so $U$ may be written in terms of a Hermitian matrix $H$

$$U = e^{iH}$$

where $H$ may be expressed in terms of nine real numbers $\alpha_i$ and $\theta$. The eight $3x3$ Gell-Mann matrices $\lambda$ are the SU(3) equivalent of the Pauli matrices in SU(2):

$$\begin{align*}
\lambda^1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\lambda^2 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\lambda^3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\lambda^4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},
\lambda^5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},
\lambda^6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},
\lambda^7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix},
\lambda^8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
\end{align*}$$ (7.23)

Note that, on the one hand, they are Hermitian and, on the other, that only $\lambda^3$ and $\lambda^8$ are diagonal Cartan operators, meaning that the rank of SU(3) is 2. The number of generators is indeed $n^2 - 1 = 8$. With these, $H$ may be expressed as

$$H = \theta 1 + \lambda \cdot a,$$

so

$$U = e^{i\theta} e^{i\lambda \cdot a}.$$

The first part is U(1) and the equation expresses $U(3) = U(1) \otimes SU(3)$. So finally we want to transform the Lagrangian in such a way that it is invariant under local SU(3) gauge transformations.

$$\psi \rightarrow S\psi, \quad \text{with} \quad S = e^{-i(q \lambda \cdot \phi(x))}.$$ In analogy with the U(1) case of (7.10), use a covariant derivative

$$D_\mu = \partial_\mu + iq \lambda \cdot A_\mu$$ (7.24)

where there are now eight gauge fields $A_\mu$, we want the transformation to function as

$$D_\mu \psi \rightarrow S(D_\mu \psi).$$

Then in the infinitesimal case, this yields a formula equivalent to (7.9)

$$A'_\mu \cong A_\mu + \partial_\mu \phi + 2q(\phi \times A_\mu)$$ (7.25)

with the cross product being

$$(B \times C) = \sum_{j,k=1}^{8} f_{ijk} B_j C_k$$ (7.26)

summed over all eight vector fields, where $f_{ijk}$ are the structure constants of SU(3). Finally, the complete Lagrangian for chromodynamics is

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - \frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} - (q \bar{\psi}\gamma^\mu \lambda \psi) \cdot A_\mu.$$ (7.27)

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This just comes from requiring that the free-particle Lagrangian be locally invariant under SU(3). The Dirac fields provide eight color currents

\[ J^\mu \equiv q(\psi \gamma^\mu \lambda \psi) \]

which constitute sources for the color fields \( A_\mu \).

We have therefore described a state of three equal-mass Dirac particles, taken to be the three color states of a given flavor of quark, interacting with eight massless vector fields, the gluons. Equation (7.27) is the correct one for the strong interaction, one for each of the six quark flavors (red, blue, green and their anti-colors).\(^{139}\)

The symmetries of this system are under rotations in the 3-dimensional color space. In general, a quark is not in a specific color state, the triplet being able to take on any direction in the space which sums the colors to white. As in the SU(2) case (7.18), now is not as simple as in QED, but contains a cross product of the vector field with itself:

\[ F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu - 2q(A^\mu \times A^\nu). \tag{7.28} \]

The cross product, defined as in (7.26), allows interactions to take place between one \( A_\mu \) field and another. Remember that we interpret field excitations as particles, via the creation and annihilation operators resulting from second quantization. So what happens here is that one field’s particle excitations may interact with another’s and “create” more particles. Such a field can be a source of itself, as in the case where a gluon may decay into a pair of gluons.\(^{140}\) So as we measure the field farther and farther away from a quark, the color charge we see actually increases. This is the inverse of the QED case, wherein virtual e^–e^+ pairs form dipoles resulting in vacuum polarization which screens the electron charge, making it seem smaller at a distance.\(^{141}\)

We can interpret this interaction among quarks as being due to the fact that they carry color. This is contrary to the case of photons, which do not carry EM charge and so do not interact with each other by the EM force.\(^{142}\)

The color force acts only between quarks and is responsible for binding them tightly one to another. But deep probes (inelastic scattering) show weaker coupling between them at higher energies, in agreement with the behavior of the preceding paragraph. Close together, they can act almost as if they were free, hence the name asymptotic freedom given to this phenomenon. It is the freedom of the quarks in the limit as the distance between them goes to zero. But as we try to pull them away from each other, the binding force becomes stronger as more gluons are created. This fact accounts for our inability to observe separate quarks.

Since SU(3) non-Cartan generators describe color-changing rotations in color space, gluons must carry color from one quark to another. Therefore each gluon must correspond to a color and an anti-color, the latter taking away what the former donates, effectively exchanging one color for another. For instance, a “blue” quark may convert to a red quark (of the same flavor) by emission of a \((b, \bar{r})\) gluon, so color is conserved.

The matrix representation of gluons has the form\(^{143}\)

\[ g^\beta_\alpha = \begin{pmatrix} r \bar{r} & r \bar{g} & r \bar{b} \\ g \bar{r} & g \bar{g} & g \bar{b} \\ b \bar{r} & b \bar{g} & b \bar{b} \end{pmatrix}, \]

so each gluon is bicolored, with one positive unit of color and one negative unit. Consider then an \( r \bar{g} \) gluon.\(^{144}\)

\(^{139}\) All this derivation based on – almost copied from – Griffiths, EP, 366-369. Essentially the same derivation, some constants aside, is done by Lancaster & Blundell, 242-244, 307-308.

\(^{140}\) For details, not all of which I can claim to understand, see Lancaster & Blundell, 428-429.

\(^{141}\) Griffiths, 68-69.

\(^{142}\) They might do so at very high energies by mutating into an electron-positron pair. I don’t think this has been observed yet.

\(^{143}\) Robinson, op. cit., 265

\(^{144}\) Example from Robinson, 254.
and three quarks

\[ g_r^q = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]

Then the possible interactions are

\[ g_r^q q_r = 0, \quad g_r^q q_g = q_r, \quad g_r^q q_b = 0, \]

so the anti-green gluon only interacts with a green quark, which it converts to a red one.

### 7.10. Physical results of gauge symmetry

In all three cases of U(1), U(2) or U(3), in order to guarantee local symmetry, one can either write the interaction term or use a covariant derivative with an addition connection term. Note that for U(1) and SU(3), EM and QCD, the vector fields – photons and gluons – are indeed massless. Such is not the case for the W and Z bosons of the weak interactions, nor for an initial doublet (of one generation) of an electron and an electron neutrino, which otherwise could correspond to the SU(2) case.

Consider the general schema of what happens in the last cases.

1. Take one, two or three spinors and write the globally symmetric free-field (Dirac) Lagrangian.
2. Make the global symmetry local, which generates an extra term in the Lagrangian.
3. Avoid the extra term by using a covariant derivative with a connection which represents one or more vector particles, the gauge bosons. This requires also taking into account their transformation properties.
4. For overall symmetry and non-zero Noether current, add in the free (Proca) Lagrangian for the gauge bosons.

You now have a locally symmetric Lagrangian for the original spinors plus a number of massless gauge bosons, including the interactions terms in their minimal coupling version. It only remains to identify this system with some particles in nature and their properties.

Klauber sums up an amazing general rule for QFT:\[145\]

If we start with the free Lagrangian and require it to be locally symmetric, then it can only be so if we add to it the particular interaction term(s) that actually describe(s) interactions in the real world.

He adds that local symmetry is essential for renormalization: No gauge invariance, no QFT.

As Sean Carroll points out, gauge theories not only give rise to forces, but impose constraints on interactions (through conservation laws due to Noether’s theorem).\[146\]

On top of all that, gauge invariance of the QED Lagrangian requires the photon to have zero mass.

But there is more...

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145 Klauber, 296.
146 Carroll, video, gauge theory. https://www.preposterousuniverse.com/blog/2020/06/30/the-biggest-ideas-in-the-universe-15-gauge-theory/. Carroll seems to be saying that multiple quarks are created, not gluons, but I doubt that.
7.11. Spontaneous symmetric breaking and the Higgs mechanism

A magnet heated above a certain critical temperature, $T_c$, loses its magnetism as the individually moving dipoles come to point randomly in different directions, making the system symmetric: Turning it through any angle changes nothing observable. As the magnet cools back down through $T_c$, a phase transition takes place and it becomes re-magnetized with the dipoles eventually all pointing in only one direction. Cooling, by forcing the dipoles to align, has caused the system to lose its symmetry, or to “hide” it. This kind of phenomenon is referred to as spontaneous symmetry breaking.\(^{147}\)

A similar event is supposed to have occurred around $10^{-35}$ seconds after the Big Bang, when a symmetric state of the Universe went through a phase change into a configuration which was no longer symmetric.

In order to show how this can take place, we must modify our Lagrangian. The requirement that it be a scalar excludes odd powers of the field, so the next simplest equation to the Klein-Gordon Lagrangian for scalar fields is obtained by addition of a 4th-power term:

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4$$

This Lagrangian has been used in the analysis of ferromagnetism and is known as the $\phi^4$ theory, or phi-4 theory. If the mass of the magnetic elements varies with temperature, we can suppose that, near the critical temperature $T_C$, a Taylor series will give $m^2 T = c(T - T_C)$ for some constant $c$.\(^{148}\) Then for $T > T_C$ the mass term in $\phi^2$ has the sign of an ordinary scalar Lagrangian. However, below $T_C$, the “mass” term is negative, corresponding to an imaginary mass, and so looks like part of a potential. In this case, the extremum at $\phi = 0$ is a local maximum and is unstable. This motivates changing the sign of the mass term. Then the minimum of the potential is not at $\phi = 0$, but at $\phi = \pm \sqrt{\frac{6m^2}{\lambda}}$. Expanding the Lagrangian around one of these values leads to a term for a real value of mass. Following these considerations, we will use a complex version of the same $\phi^4$ theory with an inverted sign of the mass term.\(^{149}\)

Remember that SU(2) gauge invariance for a doublet of two equal-mass Dirac spinors (section 7.8)? It requires the existence of three massless vector fields, but no such particles are known, nor are the initial equal-mass spinors. Here is where symmetry breaking comes to the rescue.

In order to get the idea of what happens, consider a simpler two-scalar configuration and U(1) symmetry, which is Abelian.\(^{150}\) We will complexify the system by taking linear combinations of the two fields, $\phi_1$ and $\phi_2$.

$$\phi \equiv \phi_1 + i\phi_2, \quad \text{so} \quad \phi^* \equiv \phi_1 - i\phi_2$$

so that

$$\phi^* \phi = \phi_1^2 + \phi_2^2.$$ 

The system now consists of a complex scalar. Then a $\phi^4$ Lagrangian with inverted mass-term sign\(^{151}\) for the two fields can be written

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^* (\partial^\mu \phi) + \frac{1}{2} \mu^2 (\phi^* \phi) - \frac{1}{4} \lambda^2 (\phi^* \phi)^2.$$  \hspace{1cm} (7.30)

\(^{147}\) Example from Griffiths, 376.
\(^{148}\) Schwartz, 562.
\(^{149}\) That was a hand-waving defense of equation (7.30). Another defense is the fact that it works.
\(^{150}\) Blundell and Lancaster, 430-431, treat the same problem with a non-Abelian field.
\(^{151}\) Grabbed out of a convenient hat...because it works. But we see only in a moment that it represents a scalar. Defended by Schwartz, 562.
The inverted sign on the mass ($\mu^2$) term looks like the mass is imaginary, which is not physical.\textsuperscript{152} Or, as already mentioned, one can say it looks like part of the potential energy. Now global rotational symmetry of this system is U(1) symmetry, which is Abelian. The minimum of the potential, the two middle terms, is not at $\phi = 0$, but on a circle at

$$\phi_{1\text{min}}^2 + \phi_{2\text{min}}^2 = \mu^2 / \lambda^2,$$

corresponding to the inside of the brim of the sombrero in the image, the so-called “Mexican hat potential”. (It has also been, more accurately in my opinion, compared to the bottom of a wine bottle.\textsuperscript{153}) Since any point on the circle represents the same vacuum energy, the vacuum is degenerate.

![Figure 4: “Mexican hat potential”, from Free Thought Blogs\textsuperscript{154}](image)

We are in a situation like that of the magnet at $T > T_c$. In case of a “phase transition”, the system can fall into any state on this circle – but only one. The system still possesses symmetry, but it is obliged to choose only one state and this selection of one state spontaneously breaks the symmetry.

Choice of a state of minimum, or vacuum, potential puts the system at a specific point on the inverted brim of the hat, where a particle can move in one of two perpendicular directions:

- perpendicular to the rim, in which case it must roll uphill, against a force (the gradient of the potential), thus behaving as if it had mass;
- along the rim, in which case the potential does not change and the lack of any force on the particle is interpreted as it’s being massless.

Global U(1) symmetry allows us to choose a gauge in which the vacuum is at some real value of $\phi$, $\Phi$.\textsuperscript{155} Nevertheless, the new field will be complex, depending on two real parameters:

$$\phi = \Phi + \alpha + i\beta \quad \text{and} \quad \phi^\dagger = \Phi + \alpha - i\beta.$$

Expanding the Lagrangian leads to

$$\mathcal{L} = \left[ -\frac{1}{2} \partial^\mu \alpha \partial_\mu \alpha - \frac{1}{2} 4\lambda \Phi^2 \alpha^2 - \frac{1}{2} \partial^\mu \beta \partial_\mu \beta \right] + \text{interaction terms.}$$

This now looks like a real massive scalar $\alpha$, with $m = \sqrt{4\lambda^2 \Phi^2}$, plus a massless real scalar field $\beta$, plus some interaction terms. This is a general result called Goldstone’s theorem: Global symmetry breaking always leads

\textsuperscript{152} Griffiths, 373.
\textsuperscript{153} I have much more experience with wine bottles than with sombreros.
\textsuperscript{155} This example from Robinson, 236-8.
to the existence of a massless boson, called a **Goldstone boson**.

That was for breaking of global symmetry. In order for the Lagrangian to be invariant under *local* U(1) gauge transformations

\[ \phi \rightarrow e^{i\theta(x)} \phi, \quad (7.31) \]

we must employ the usual trick of introducing a massless vector gauge field \( A^\mu \) and a covariant derivative \( D_\mu \), such that\(^{156}\)

\[ D_\mu = \partial_\mu + iq A_\mu \quad (7.32) \]

and

\[ \mathcal{L} = \frac{1}{2} (D_\mu \phi^* (D^\mu \phi) + \frac{1}{2} \mu^2 (\phi^* \phi) - \frac{1}{4} \lambda^2 (\phi^* \phi)^2 - \frac{1}{16\pi} F^\mu\nu F_{\mu\nu}. \quad (7.33) \]

Let’s choose a particular state, thus breaking the symmetry, by changing coordinates so that

\[ \eta \equiv \phi_1 - \frac{\mu}{\lambda} \quad \text{and} \quad \xi \equiv \phi_2. \quad (7.34) \]

Then the Lagrangian expands to\(^{157}\)

\[ \mathcal{L} = \left[ \frac{1}{2} (\partial_\mu \eta)(\partial^\mu \eta) - \mu^2 \eta^2 \right] + \left[ \frac{1}{2} (\partial_\mu \xi)(\partial^\mu \xi) \right] + \left[ - \frac{1}{16\pi} F^\mu\nu F_{\mu\nu} + \frac{1}{2} \left( \frac{q\mu}{\lambda} \right)^2 A_\mu A^\mu \right] + \ldots + \left( \frac{q\mu}{\lambda} (\partial_\mu \xi) A^\mu \right). \quad (7.35) \]

This Lagrangian represents the same state as (7.33), only now it contains the following terms (separated by brackets):

- One term which looks like the K-G Lagrangian for a scalar particle \( \eta \) of mass \( \mu \). (Spoiler: This is the Higgs field.)
- A second term which corresponds to a massless scalar Goldstone boson.
- A Proca term for the vector field \( A^\mu \) which – lo, behold! – now has a mass

\[ m_A = 2\sqrt{\frac{q\mu}{\lambda}}, \]

which is proportional to the shift \( \frac{\mu}{\lambda} \) in \( \phi_1 \) and so is also due to the potential minimum’s not being at symmetric zero.
- An extraordinarily messy term coupling the three fields \( \eta, \xi \) and \( A^\mu \).
- A term in \( (\partial_\mu \xi) A^\mu \), which looks inconveniently like a \( \xi \) turning into an \( A^\mu \).

So we can say that local U(1) symmetry and subsequent symmetry breaking by the translation of the field zero, which we will see is its vacuum state, to the minimum of the potential has conferred mass on two fields, scalar and vector, and produced a third – unwanted – one.

Both these problem terms involve \( \xi = \phi_2 \). That can be fixed. We can specify a **particular gauge** for the U(1) transformation (7.31) so that\(^{158}\)

\[ \theta = -\tan^{-1}\left( \frac{\phi_2}{\phi_1} \right). \quad (7.36) \]

Then

---

156 Griffiths, 378-80.
157 Griffiths, 379.
158 Robinson, 238-9, applies such a unitary gauge earlier, which simplifies the series of equations. A different path yet is taken by Blundell and Lancaster, 239-242.
\[ \phi \rightarrow e^{i\theta} \phi = (\cos \theta + i \sin \theta)(\phi_1 + i \phi_2) \]

becomes real and \( \xi = 0 \), which eliminates both the Goldstone boson and the unfortunate \( \xi - A^\mu \) interaction term. Finally, we have

\[
\mathcal{L} = \left[ \frac{1}{2} (\partial_\mu \eta)(\partial^\mu \eta) - \mu^2 \eta^2 \right] + \left[ -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} \left( \frac{\mu}{\lambda} \right)^2 A_\mu A^\mu \right] + \text{interaction terms and a constant.}
\]

(7.37)

We started with one complex scalar field and a massless vector field. From that, local symmetry breaking has led to a single, massive (vector) gauge field \( A^\mu \) and a single massive scalar field \( \eta \), called the Higgs field. The mechanism whereby the scalar field is conferred a mass by shifting the field zero is called the Higgs mechanism.

Recap:

- Break symmetry by expressing the Lagrangian relative to the vacuum, at a non-zero minimum value of the potential.
- Choose a gauge to eliminate Goldstone bosons and give mass to a scalar.

So we have made the following discoveries:

- Breaking a global symmetry always leads to the creation of a massless Goldstone boson.
- Local symmetry requires the presence of a gauge field, and breaking the symmetry adds mass to that field.

We have seen this in the case of global and local U(1)-invariant Lagrangians. Since the quantities involved are scalars, they commute and this is an Abelian gauge theory.

Note well that in equations (7.31) through (7.37), the force-carrying vector boson field \( A_\mu \) is introduced in order to assure local U(1) symmetry. The Higgs only shows up as a result of symmetry breaking. Sabine Hossenfelder claims that, for this reason, most physicists don’t call the Higgs a force. “The reason is that the exchange particles of electromagnetism, the strong and weak nuclear force, and even gravity, hypothetically, all come out of symmetry requirements. The Higgs-boson [sic] doesn’t. That may not be a particularly good reason to not call it a force carrier, but that’s the common terminology. Four fundamental forces, among them is gravity, which isn’t a force, but not the Higgs-exchange, which is a force. Yes, it’s confusing.”

We shall see shortly that the Feynman method of adding possible states is a perturbation theory around a ground state (or vacuum) which is a state of minimum energy, hence the necessity of a choice of vacuum (7.34) which brings about spontaneous symmetry breaking. Then a suitable gauge choice (7.36) causes the gauge field to “eat” the Goldstone boson and gain a mass. At the same time, it acquires a third polarization state, whereas massless particles only have two. So we can say:

Local gauge invariance + spontaneous symmetric breaking \( \implies \) Higgs mechanism.

This is the process which is supposed to have taken place when the Universe was very, very young and which brought about creation of mass for the weak-interaction gauge bosons \( W^\pm \) and \( Z^0 \).

The Higgs mechanism is responsible for the masses not only of vector bosons but of fermions (e, \( \tau \), …) and even quarks. A more nuanced statement of this is “…the Higgs field is indirectly responsible for the fermion masses.”

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159 Griffiths, 380.
161 Griffiths, 380-1.

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masses... It just allows you to have consistent extra terms in your initial Lagrangian, which would give you the mass terms result..."\textsuperscript{162} Proton and neutron masses, though, come not from Higgs but from quark binding energy.

For once, Wikipedia explains it simply and understandably:

"The simplest description of the [Higgs] mechanism adds a quantum field (the Higgs field) that permeates all space to the Standard Model. Below some extremely high temperature, the field causes spontaneous symmetry breaking during interactions. The breaking of symmetry triggers the Higgs mechanism, causing the bosons it interacts with to have mass."\textsuperscript{163}

7.12. Electroweak interactions

We have now studied the mechanisms of the EM and strong forces, but have yet to understand the weak force. The SU(2) doublet Lagrangian (7.22) alone is inadequate to explain it, since the measured masses of the two candidate particles, \(e^-\) and \(\nu_e\), are quite different and the gauge bosons are not massless. It is proposed that when the Universe was less than \(10^{-12}\) seconds of age, there was only an electroweak force (plus gravity) which unified both the weak force and the EM force. This force would have been symmetric before a phase change resulted in "spontaneous" symmetry breaking and bestowed masses on the gauge bosons.

First, a word on groups in the Standard Model. The SM is considered to be a gauge theory with gauge (symmetry) group

\[
SU(3)_C \times SU(2)_L \times U(1)_Y,
\]

where \(C\) stands for color; \(L\) means it acts on left-handed states only; and \(Y\) stands for hypercharge. The hypercharge \(U(1)_Y\) is not the same as that, \(U(1)_{EM}\), for EM, which we have studied.\textsuperscript{164} This will become clearer in due course. The gauge bosons implied by this product of groups is thus 8 from \(SU(3)_C\), 3 from \(SU(2)_L\) and one from \(U(1)_Y\), the only one of these groups which is Abelian. A group state – representation and charge – can then be indicated by specifying a triplet of numbers. For instance,

\((1, 2, -\frac{1}{2})\) indicates an \(SU(3)_C\) singlet (which we can ignore) and an \(SU(2)_L\) doublet with hypercharge \(-\frac{1}{2}\).

In order to investigate the electroweak force, we adopt the state \((1, 2, -\frac{1}{2})\), which means we can ignore the \(SU(3)_C\) part and consider only the \((2, -\frac{1}{2})\) part. We then proceed by analogy from (7.10) and (7.19) for \(U(1)_Y\) and SU(2), and write a covariant derivative for the two at once as follows:\textsuperscript{165}

\[
(D_\mu \phi)_i = \partial_\mu \phi_i - ig_2 W^a_\mu T^a_2 + g_1 B_\mu Y |i, j, \phi_j.
\]

where the generators of the fundamental, 2-d representation of SU(2) are written in terms of the Pauli matrices as

\[
T^a = \frac{1}{2} \sigma^a
\]

and the generator of \(U(1)\) as

\[
Y = C \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}.
\]

\(C\) being the hypercharge. This is the hypercharge associated with the high-energy \(U(1)_Y\) symmetry. The \(SU(2)_L\) gauge fields are conventionally written as \(W^a_\mu\) and the \(U(1)_Y\) gauge field as \(B_\mu\). There are now two

charges, one for each group, and two coupling constants in the covariant derivative, \( g_2 \) for \( SU(2)_L \) and \( g_1 \) for \( U(1)_Y \). Putting in the generators, we find the bracketed term in the covariant derivative (7.38) to be

\[
g_2 W^a T^a + g_1 B_\mu Y = \frac{1}{2} \left( g_2 W_\mu^1 - g_1 B_\mu \quad g_1 (W_\mu^1 - iW_\mu^2) \right)
\]

To solve this, we will expand the Pauli matrices and then, in order to build a Lagrangian for a complex scalar doublet, suppose again a \( \phi^4 \) potential like (7.30).

\[
V(\phi^+, \phi) = \frac{1}{4} \lambda (\phi^+ \phi - \frac{1}{2} v^2).
\]

We then break the symmetry by choosing a minimum for the potential (Higgs mechanism), as was done in section 7.11. We also have to pick a gauge, in this case the unitary gauge, in order to assure that the first component of the field be real and the second zero.\(^{166}\) The vacuum expectation value (VEV) of the field is then

\[
\langle 0 | \phi^+ | 0 \rangle = \langle \phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} u \\ 0 \end{pmatrix}
\]

Noting that \( V = 0 \) at the VEV, we define two fairly obvious linear combinations \( W_\mu^\pm \) and write the VEV Lagrangian mass term as

\[
\mathcal{L}_{\langle \phi \rangle} = -\frac{1}{8} g_2^2 v^2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{c_W} Z_\mu \sqrt{2}W^+_\mu \\ \sqrt{2}W^-_\mu \end{pmatrix}^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]

\[
= -\left( \frac{g_2 v}{2} \right)^2 W^{+\mu} W^{-\mu} - \frac{1}{2} \left( \frac{g_2 v}{2c_W} \right)^2 Z^\mu Z_\mu.
\]

where we ignore the fourth term in the matrix because it doesn’t survive the matrix multiplication. We can identify the W and Z masses from this term as\(^{167}\)

\[
M_W = \frac{g_2 v}{2} \quad \text{and} \quad M_Z = \frac{M_W}{c_W} = \frac{g_2 v}{2c_W} = \frac{v}{2} \sqrt{g_1^2 + g_2^2}.
\]

The solution, only reached after a fair amount of math, expresses the physical particles as sums of the W and B fields.

\[
W^{+}_\mu = \frac{1}{\sqrt{2}} (W^1_\mu - iW^2_\mu), \quad W^{-}_\mu = \frac{1}{\sqrt{2}} (W^1_\mu + iW^2_\mu),
\]

\[
Z_\mu = c_W W^3_\mu - s_W B_\mu, \quad A_\mu = s_W W^3_\mu + c_W B_\mu,
\]

which make use of a shorthand notation in terms of something called the weak mixing angle:

\[
\theta_W = \tan^{-1} \left( \frac{g_1}{g_2} \right),
\]

\[
s_W = \sin \theta_W = \frac{g_1}{\sqrt{g_1^2 + g_2^2}}, \quad c_W = \cos \theta_W = \frac{g_2}{\sqrt{g_1^2 + g_2^2}}.
\]

So, once again, symmetry breaking has contributed a mass to the W and Z fields but leaves the A field massless, as desired. The W and Z vector bosons are the force-carrying fields of the weak force.

Again, the \( W^{\pm} \) are sums of non-Cartan generators of SU(2) and so are ladder operators, capable of raising or

---

\(^{166}\) Robinson, 252-259; Schwichtenberg, 150-157.

\(^{167}\) Yes, we are getting a bit ahead of ourselves here. Just assume these definitions, or wait for the solution.
lowering the charge. They constitute what is referred to as a **weak charged current**, responsible for events like the decay of a neutron into a proton, an electron and an anti-electron-neutrino. On the other hand, $Z^0$ and $A$ are sums of Cartan generators of SU(2) and U(1), respectively, and will not change the charge. Since the basis vectors are eigenvectors of the Cartan operators, the result of operating with them just gives the eigenvalue multiplied by the same basis vector. The $Z^0$ is thus responsible for the **weak neutral current**, accounting, for instance, for neutrino-electron scattering. Back in those very brief times, the good ol' massless EM photon was not just a descendant of the EM gauge boson but a linear combination of that and of the Cartan generator of SU(2). The $A$ is massless and a single U(1) symmetry remains unbroken. These are the field and gauge group of EM. The $W$ and $Z$ masses are quite large, making the weak force effective only over a very short range.

This result was obtained by symmetry breaking. Before the symmetry broke, there was only a Higgs complex vector field with four massless vector boson gauge fields, each behaving generally like a photon. The low-energy, “broken” theory is based on linear combinations of the original fields, three of which have gained mass. The original high-energy state is the content of **electroweak theory**. The original four photon-like fields break to form two distinct forces, the broken weak and the unbroken EM. This merging of two states into one at high energies is called **unification**.

Now let’s then expand $\phi$ around the vacuum:

$$\phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} v + h(x) \\ 0 \end{pmatrix}.$$  

The potential (7.39) then is

$$V(\phi) = \frac{1}{4} \lambda v^2 h^2 + \frac{1}{4} \lambda v H^3 + \frac{1}{16} \lambda H^4.$$  

We can read off the Higgs mass as $m_h = \frac{1}{2} \lambda v^2$, which has been found experimentally to be $125.10 \pm 0.14$.\(^{168}\)

Putting everything together, one can calculate a Lagrangian for the electroweak gauge fields and the Higgs boson in unitary gauge. The resulting five-line equation leads to the following conclusions.\(^{169}\)

- The photon field $A_\mu$ does not couple to the Higgs field, as there is no term with the product of the two.
- The $W^\pm$ fields do interact with the Higgs field, there being a term proportional to $W^+ W^- h$. This accounts for the decay of the Higgs particle into pairs of W (and Z) bosons.

### 7.13. Summary – Forces and particles

Now we can see how symmetry considerations explain forces and particles (or fields).

Valid Lagrangians possess symmetries. Indeed, what constitutes a valid Lagrangian is imposed by the requirement of Lorentz invariance -- symmetry under Lorentz transformations. Noether’s theorem tells us that for each symmetry, a calculable quantity is conserved -- energy or momentum or angular momentum, for instance, but also spin, hypercharge or color.

The notion of symmetry involves some transformation under which the system is left in a different state which is symmetric to the original one. Such transformations are seen as changes resulting from the application of forces. Since each symmetry of the Lagrangian is described by a Lie group, we can take a representation of the Lie group as describing the results of applying a particular force to the system described by the Lagrangian.

Lie groups tell how rotations, Lorentz boosts and changes of phase (unitary transformations) modify vectors within the vector space of the representation of the group.

---

\(^{169}\) Srednicki, 530; Robinson, 258-9.
A representation is described by means of a Lie algebra, which is made up of generators of a representation. For a given group, the number of generators is always the same, even though the dimensions of representations vary.

Some of the generators are diagonal and therefore correspond to eigenvectors, which can be used as a basis for the vector space of the representation. They constitute the **Cartan subalgebra** of the Lie algebra and their number is the rank of the algebra. The eigenvalues are the physically measurable charges on which the force acts, and are equal in number to the dimension of the fundamental representation, which in the case of SU(n) or SO(n) consists of $n \times n$ matrices. In other words, an arbitrary vector will have components along different eigenvectors of the Cartan generators. Transformations of the group then will rotate one vector into another, changing the eigenvalues to which it corresponds. The “rotations” may be in Minkowski spacetime or in some internal space, such as spin or color space.

An arbitrary SU(n) group always has $n^2 - 1$ generators and be of rank $n - 1$, whereas an arbitrary SO(n) group has $\frac{n(n-1)}{2}$ generators.

Let’s resume:

- A fundamental **force** – strong, weak, electromagnetic or, hopefully, gravitational – is described by a particular representation of a Lie group.
- The **eigenvectors** of the diagonal Cartan generators describe the **particles** which are acted on by the force.
- The **eigenvalues** are the physically measurable **charges** (such as spin).
- The number of **charges** fixes the **dimension** of the fundamental representation.
- The **generators** of the group, which parameterize the infinitesimal changes wrought by the transformation on the vector space of the representation, correspond to the **gauge bosons**, the force-carrying particles – photons, gluons, W and Z bosons – which act on the physical particles. There are two sorts.
  - Forces of Cartan generators carry forces which act on all charged particles in the group by transferring energy-momentum.
  - Forces of non-Cartan generators not only can transfer energy-momentum but also can constitute **creation/annihilation (ladder) operators** which change the charge.

But symmetry and QFT bring more than the understanding of forces and how they arise.

For instance, in SU(2), there are three generators, $J_{1-3}$, of which only one is a Cartan generator, by convention $J_3$. Its eigenvectors form a basis for the space. In the 2-dimensional representation, for $j = \frac{1}{2}$, the eigenvectors represent particles of positive and negative spin, $j = \pm \frac{1}{2}$. The non-Cartan generators, $J_1$ and $J_2$, are used to construct ladder (raising and lowering) operators which change the charge, in this case, the spin. (In the case of a quantized field, they change the number of particles with this value of the spin.)

In the case of the strong force, the 3-dimensional fundamental representation of SU(3) gives rise to the three eigenvalues which are the three quark colors. The gauge bosons describe how the colors are changed. The Cartan generators (photon, $Z^0$) represent force-carrying particles which can modify, say, position or momentum, but without any change in charge. The non-Cartan generators (W bosons and gluons) are those which comprise the QM raising and lowering operators, so it is not a surprise that they can also bring about changes in charge.\(^{170}\)

---

170 Robinson, 116-117.
7.14. So what’s the big deal?

Quantum field theory is at the heart of modern physics, explaining QED, QCD, the electroweak force and the Standard Model. It is based on Lagrangian mechanics and takes into account the requirement of Special Relativity that the Lagrangian must be invariant under Lorentz transformations. As we have seen, this means it must be invariant under SU(2), the covering group for both SO(3) and the restricted Lorentz group $\Lambda^3_+$. This limits possible forms for the Lagrangian, as we saw in section 5.

**Deal 1:** Invariance under Lorentz transformation leads to the form of the Lagrangian for scalar, spinor and vector particles and fields.

Applying local gauge invariance to free-field equations for various numbers of spinors leads to an interaction term, often expressed as an additional term in the covariant derivative, and to the existence of additional massless spin-1 particles or fields (bosons). This is summarized in Table 9.

<table>
<thead>
<tr>
<th>No. spinors</th>
<th>Symmetry</th>
<th>No. massless spin-1 particles required</th>
<th>Interpretation of spin-1 particles</th>
<th>QM theory concerned</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>U(1)</td>
<td>1</td>
<td>photon</td>
<td>electrodynamics</td>
</tr>
<tr>
<td>2</td>
<td>SU(2)</td>
<td>3</td>
<td>vector bosons, but ...</td>
<td>weak interactions</td>
</tr>
<tr>
<td>3</td>
<td>SU(3)</td>
<td>8</td>
<td>gluons</td>
<td>chromodynamics</td>
</tr>
</tbody>
</table>

Table 9. Effects of SU(n) transformations on spinors.

**Deal 2:** Invariance under U(1) and SU(3) leads to EM and QCD.

U(1) and SU(3) symmetries thus give rise to photons and gluons, the massless vector “particles” of EM and strong interactions, respectively. In the case of the weak interactions, the W and Z particles concerned are most definitely not massless, so we must draw on “spontaneous” symmetry breaking\(^{171}\) and the Higgs mechanism.

**Deal 3:** Invariance under SU(2) and Y(1)\(^{\nu}\) leads to the electroweak theory.

In all three deals (cases), spontaneous symmetry breaking is needed in order to bestow masses on the gauge particles (except the photon).

7.15. Construction of SU(3) multiplets – the Eightfold Way

SU(3) has $(n^2 - 1) = 8$ parameters and so 8 generators in every representation, whatever be the dimension of the representation.\(^{172}\) In its fundamental representation of 3x3 matrices, the standard generators are taken to be $\frac{1}{2}$ the Gell-Mann matrices, $\lambda^i$, $i = 1 - 8$:

$$
\lambda^1 = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\lambda^2 = \begin{pmatrix}
0 & -i & 0 \\
i & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\lambda^3 = \begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 0
\end{pmatrix}
$$

\(^{171}\) I object to the word “spontaneous” here, which indicates the symmetry breaking is not brought about by an external agency, to be confusing to the point of being incorrect. But I have no better candidate.

\(^{172}\) Robinson, 111-116.
Only two of the generators are diagonal and so Cartan generators, $\lambda^3$ and $\lambda^8$, meaning that the rank of SU(3) is 2. All the generators are Hermitian.

The upper-left quadrangles of $\lambda_1 - \lambda_3$ are the Pauli matrices. So the first three matrices are a closed SU(2) subgroup of SU(3).

Being a 3-d representation, this one has three eigenvectors, each one with a pair of eigenvalues, which in turn means three basis vectors.

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$  

(We could perhaps better call them u, d and s without waiting, but....) Pairs of eigenvalues for $\lambda^3$ and $\lambda^8$ can be represented by so-called weight vectors, column doublets in this case, of pairs of eigenvalues for a given eigenvector. They are

$$t_1 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2\sqrt{3}} \end{pmatrix}, \quad t_2 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2\sqrt{3}} \end{pmatrix} \quad \text{and} \quad t_3 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{3}} \end{pmatrix}. \quad \text{(7.44)}$$

corresponding to the basis vectors $v_1$, $v_2$ and $v_3$.

Remembering the happy results of defining ladder operators in SU(2) and SO(3), let’s try this for SU(3). First, define the generators by

$$F_i = \frac{1}{2} \lambda_i. \quad \text{(7.45)}$$

Then take what is called the spherical representation of the F operators.\(^{174}\)

$$T_\pm = F_1 \pm iF_2, \quad T_3 = F_3,$$
$$V_\pm = F_4 \pm iF_5,$$
$$U_\pm = F_6 \pm iF_7, \quad Y = \frac{2}{\sqrt{3}}F^8. \quad \text{(7.46)}$$

One can work out the commutation properties of these operators, including definitions of $U_3$ and $V_3$ by analogy with $T_3$:\(^{175}\)

$$[T_3, T_\pm] = \pm T_\pm, \quad [T_+, T_-] = 2T_3,$$
$$[T_3, U_\pm] = \mp \frac{1}{2} U_\pm, \quad [U_+, U_-] = \frac{3}{2} Y - T_3 \equiv 2U_3,$$
$$[T_3, V_\pm] = \mp \frac{1}{2} V_\pm, \quad [V_+, V_-] = \frac{3}{2} Y + T_3 \equiv 2V_3. \quad \text{(7.47)}$$

We have already noted that the T’s form a closed SU(2) subset. The first line of (7.47) confirms that the operators

---

173 Apologies for not distinguishing carefully between superscripts and subscripts in this context.
174 I am ignoring the hats which should adorn the F and $\lambda$ operators.
175 Greiner, 203-204, 212-215.
$T_i (i = +, -, 3)$ are consistent with the Lie algebra for SU(2) of (4.15). The same thing can be shown to be true for the $U$ and $V$ operators:

$$[U_3, U_\pm] = \pm U_\pm, \quad [U_+, U_-] = 2U_3$$
$$[V_3, V_\pm] = \pm V_\pm, \quad [V_+, V_-] = 2V_3.$$

All three subalgebras of SU(3) – call them T-spin, U-spin and V-spin – are isomorphic to SU(2). (The whole thing is sometimes referred to as F-spin.) Better yet, $T_\pm$, $U_\pm$ and $V_\pm$ are ladder operators. Call our two Cartan operators $T_3$ and $Y$, and our eigenvectors $|t_3, y\rangle$, so that (denoting eigenvectors by lower case)

$$T_3 |t_3, y\rangle = t_3 |t_3, y\rangle, \quad Y |t_3, y\rangle = y |t_3, y\rangle.$$

Then from (7.47)

$$(T_3 V_\pm - V_\pm T_3) |t_3, y\rangle = (T_3 V_\pm - t_3 V_\pm) |t_3, y\rangle = \pm \frac{1}{2} V_\pm |t_3, y\rangle,$$

by (7.47) tells us that

$$T_3 V_\pm |t_3, y\rangle = (t_3 \pm \frac{1}{2}) V_\pm |t_3, y\rangle,$$

i.e., $V_\pm$ is a couple of ladder operators which raise/lower $t_3$ by $\frac{1}{2}$. In a similar manner, one can show that $U_\pm$ lowers/raises $t_3$ by $\frac{1}{2}$, i.e., in the opposite direction to $V_\pm$. Then

$$[Y, V_\pm] = \pm V_\pm$$

may be used to show that

$$U_\pm (V_\pm |t_3, y\rangle) = (y \pm 1) (V_\pm |t_3, y\rangle),$$

so $V_\pm$ raises/lowers $y$ by 1. The same is true for $U_\pm$.

So, more succinctly, including as-yet uncalculated normalization constants:

$$T_\pm |t_3, y\rangle = \alpha |t_3 \pm 1, y\rangle,$$
$$U_\pm |t_3, y\rangle = \beta |t_3 \mp \frac{1}{2}, y \pm 1\rangle,$$
$$V_\pm |t_3, y\rangle = \gamma |t_3 \pm \frac{1}{2}, y \pm 1\rangle. \quad (7.48)$$

From these equations, we can diagram the actions of the ladder operators in the $t_3 - Y$ plane.

176 We are also not worrying about normalization factors here.
Now we can understand some interesting points.

The SU(3) algebra includes the three T, U and V subalgebras, each of which is isomorphic to the algebra of SU(2). So SU(3) multiplets can be constructed from coupled T, U and V multiplets.

Since $U_3$ and $V_3$ are linear combinations of $T_3$ and $Y$ by (7.47), all four may be simultaneously diagonalized, with eigenvalues $t_3$, $y$, $U_3 = \frac{1}{2}(\frac{3}{2}Y - T_3)$ and $V_3 = \frac{1}{2}(\frac{3}{2}Y + T_3)$.

The ladder operators shift the SU(3) multiplet states as in Figure 5, where the operator end-points are on a regular hexagon. So an SU(3) multiplet will be constructed from a T multiplet parallel to the $T_3$ axis and U and V multiplets parallel to the U and V axes in Figure 5. They are coupled by the non-zero commutation relations among them, e.g., $[T_+, V_-] = -U_-$. The symmetry among the T, U and V algebras requires symmetry with respect to the $T_3 = 0$, $U_3 = 0 = \frac{3}{2}(Y - T_3)$ and $V_3 = 0 = \frac{3}{2}(Y + T_3)$ with an angle of 120° between any two. Normally, the SU(3) multiplet is centered on the origin $T_3 = Y = 0$. Further symmetry considerations show that an SU(3) multiplet must be a regular triangular or hexagonal structure.

Within the F-spin algebra, the T, U and V-spin subalgebras appear symmetrically, so the simplest non-trivial SU(3) multiplet will contain doublets of all three. This leads us to the diagrams of Figure 6. Each diagram contains an isodoublet $T = \frac{1}{2}$ and an isosinglet $T = 0$. For the representation $[3]$, the isodoublet is composed of the states $\psi_1 = |\frac{1}{2}, y\rangle$ and $\psi_2 = |-\frac{1}{2}, y\rangle$ and the isosinglet $\psi_3 = |0, y\rangle$.

The values of the hypercharge $Y$ may now be determined. From the diagram, $\psi_2$ and $\psi_3$ form the U-spin doublet, so the singlet must be $\psi_1$. $U_3 \psi_1 = 0$.  

---

177 Greiner, 212.
178 This overview of the subject must be qualified as “quick and dirty”. For a more compete version, see Greiner, 231-3.
Then the definitions of $U_3$ and $V_3$ may be used to find the eigenvalues of $Y$:

$$Y\psi_1 = \frac{1}{3}\psi_1, \quad Y\psi_2 = \frac{1}{3}\psi_2, \quad Y\psi_3 = -\frac{2}{3}\psi_3$$

and the corresponding negative values for the antiparticles. The charge is given by the Gell-Mann-Nishijima formula

$$q = t_3 + \frac{1}{2} y.$$  \hspace{1cm} (7.49)

The charges are then

$$Q\psi_1 = \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{3} = 2, \quad Q\psi_2 = -\frac{1}{2} + \frac{1}{2} \cdot \frac{1}{3} = -\frac{1}{3}, \quad Q\psi_3 = 0 + \frac{1}{2} \cdot (-\frac{2}{3}) = -\frac{1}{3}.$$  

So the representation is of three particles with fractional charges – the **quarks** (i.e., the lightest three quarks) $u$, $d$ and $s$ ($\psi_1$, $\psi_2$ and $\psi_3$).

![Figure 6](Image)

Figure 6. The smallest non-trivial representation of SU(3)

The values for $[3]$ are the negative of these, those of the corresponding antiquarks. It is clear from the eigenvalues that the $[3]$ diagram represents particles, the $[\bar{3}]$ diagram, antiparticles. In support of this, the Gell-mann-Nishijima formula shows that

$$Q\bar{\psi}_\nu = q_\nu \psi_\nu = q_\nu \psi_\nu.$$  

They are the two fundamental representations of SU(3). "In principle the construction of all irreducible representations of the [sic] SU(3) requires only one of the two fundamental representations $[3]$ and $[\bar{3}]$. ... However, for reasons of physics, one needs both fundamental representations, because quarks (represented by $[3]$) and antiquarks (represented by $[\bar{3}]$) differ by their baryon number ($B=1/3$ for quarks, $B=-1/3$ for antiquarks) and charge."179 With these baryon numbers, the Gell-mann-Nishijima relation works out consistently for the charges.

Remember that SU(2) representations (multiplets) can be constructed from the fact that each corresponds to a value $j = 0, \frac{1}{2}, 1, \frac{3}{2}, ...$ with dimension $(2j+1)$ and states $|jm\rangle = +j, ... -j$. Alternatively, they could be constructed by successive coupling of the fundamental doublets $|j\rangle = |\frac{1}{2}, m\rangle = \frac{1}{2}, \frac{1}{2}$, $m = \frac{1}{2} \pm \frac{1}{2}$. This latter method will be used to construct SU(3) multiplets from the two fundamental multiplets $[3]$ and $[\bar{3}]$.

---

179 Greiner, 242.
An SU(3) multiplet expressed as a combination of p quarks and q antiquarks is denoted D(p,q). A state deemed of maximal weight is constructed starting with p quarks of maximal weight state \( \frac{1}{2}, \frac{1}{3} \) and q antiquarks of maximal weight state \( \frac{1}{2}, -\frac{1}{3} \), so the state has the following values of \( T_3 \) and \( Y \) for this maximal state:

\[
(T_3)_{\text{max}} = \frac{p + q}{2} \quad \text{and} \quad (Y)_{\text{max}} = \frac{p - q}{3}.
\] (7.50)

This state will be the farthest to the right on the diagram. Other states are then constructed from it by use of the ladder (shift) operators \( V_{\pm} \) and \( U_{\pm} \). Then the states \([3]\) and \([\bar{3}]\) of Figure 6 are the single-quark state D(1,0) and the single antiquark state D(0,1).

![Diagram of the baryon decuplet](image)

Figure 7. The baryon decuplet.

Take the example of the baryon decuplet of Figure 7. Using the D(3,0) representation, we know from (7.50) that the maximum point is at \( \left( \frac{3}{2}, 1 \right) \) with charge \( q=+2 \), from (7.49). It therefore may correspond to the \( \Delta^{+++} \), composed of three up quarks, uuu. Now use \( V_{-} \) to construct the other three baryons on the right hand side, at each step reducing \( t_3 \) and \( Y \) by \( \frac{1}{2} \) and 1, respectively. This is equivalent to transforming an up quark to a strange one at each step and gives us particles successively at \( \left( 1, 0 \right), \left( \frac{1}{2}, -1 \right) \) and \( \left( 0, -2 \right) \). From each of these three, we may use \( U_{+} \) change the coordinates each time by \( \left( -\frac{2}{3}, 1 \right) \), changing a strange quark to a down one each time, and this will generate the other particles. Or we could use \( T_{-} \), to get the equivalent diagram by changing up quarks one at a time to down ones.

These diagrams are representative of what is called the **Eightfold Way** of particle physics.

The meson octet of Figure 7 is the D(1,1)=[8] representations formed of a quark and antiquark. These are the correct components of baryons and mesons.

---

180 This is, again, a quick version of what is explained in more detail by Greiner, 241-6.
The baryon octet of Figure 9 is more complicated and necessitates a digression.

QFT proves\(^{181}\) that bosons must have symmetric wave functions and fermions, antisymmetric. This of course leads to the Exclusion Principle – on which the existence of chemistry and so everything else is based. The symmetry of a quark wave function has four parts – space, spin, flavor and color. The law requiring hadrons to be color neutral is in fact an example of a more fundamental law which states that “every naturally occurring particle is a color singlet.”\(^{182}\) So the color part is antisymmetric, which means the rest must be symmetric. The ground state we are assuming must be spatially symmetric, so the product of the spin and flavor parts must be symmetric. The 27 (3\(^3\)) possible combinations of three quarks can be grouped into symmetric, antisymmetric or mixed states – a completely symmetric decuplet, a completely asymmetric singlet and two octets of mixed symmetry (antisymmetric in either particles 1 and 2 or 2 and 3).\(^{183}\) In group-theory speak, we can see this as the decomposition of the product of three fundamental states

\[
3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1.
\]

Now the baryon decuplet is composed of completely symmetric states, since, e.g., what is called \(\Delta^+\) in Figure 7 can be considered as a symmetric combination \((uud + udu + duu) \sqrt{3}\), and so on for the other states. So this decuplet must go along with a symmetric spin state. The baryon octet is asymmetric in two quarks, say 1 and 2. But the corner quarks would be uuu, ddd and sss, all highly symmetric. So they must go. Such is a somewhat hand-wavy argument for why the baryon octet looks like a triply truncated triangle.

---

181 Where?
182 Griffiths, 187.
183 Griffiths, 185-6.
Finally, here are the most common particles, with their spin, quark composition and charge.

<table>
<thead>
<tr>
<th>Baryons</th>
<th>Spin</th>
<th>Particle</th>
<th>Quark content</th>
<th>Charge</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2</td>
<td></td>
<td>p</td>
<td>uud</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>n</td>
<td>udd</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Λ</td>
<td>uds</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Σ⁺</td>
<td>uus</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Σ⁰</td>
<td>uds</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Σ⁻</td>
<td>dds</td>
<td>-1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Ξ⁰</td>
<td>uss</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Ξ⁻</td>
<td>dss</td>
<td>-1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Λ⁺</td>
<td>udc</td>
<td>1</td>
</tr>
<tr>
<td>3/2</td>
<td></td>
<td>Δ</td>
<td>uuu, uud, udd, ddd</td>
<td>2, 1, 0, -1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Σ⁺*</td>
<td>uus, uds, dds</td>
<td>1, 0, -1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Ξ⁺*</td>
<td>uss, dss</td>
<td>0, -1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Ω⁻</td>
<td>sss</td>
<td>-1</td>
</tr>
</tbody>
</table>

Table 10. Properties of most common baryons
### Table 11. Properties of common mesons

<table>
<thead>
<tr>
<th>Spin</th>
<th>Particle</th>
<th>Quark content</th>
<th>Charge</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(\pi^\pm)</td>
<td>0, 1, -1</td>
<td>0, 1, -1</td>
</tr>
<tr>
<td></td>
<td>(\pi^0)</td>
<td>(u(\bar{u}) - d(\bar{d}))/(\sqrt{2})</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>(K^\pm)</td>
<td>us, s(\bar{u})</td>
<td>1, -1</td>
</tr>
<tr>
<td></td>
<td>(K^0, \bar{K}^0)</td>
<td>ds, s(\bar{d})</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>(\eta)</td>
<td>(u(\bar{u}) + d(\bar{d}) - 2s(\bar{s}))/(\sqrt{6})</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>(\eta')</td>
<td>(u(\bar{u}) + d(\bar{d}) + s(\bar{s}))/(\sqrt{3})</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>(D^\pm)</td>
<td>cd, d(\bar{c})</td>
<td>1, -1</td>
</tr>
<tr>
<td></td>
<td>(D^0, \bar{D}^0)</td>
<td>c(\bar{u}), u(\bar{c})</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>(D_s^\pm)</td>
<td>cs, s(\bar{c})</td>
<td>1, -1</td>
</tr>
<tr>
<td></td>
<td>(B^\pm)</td>
<td>u(\bar{b}), b(\bar{u})</td>
<td>1, -1</td>
</tr>
<tr>
<td></td>
<td>(B^0, \bar{B}^0)</td>
<td>d(\bar{b}), b(\bar{d})</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>(\rho)</td>
<td>ud, (uu - d(\bar{d}))/(\sqrt{2}), d(\bar{u})</td>
<td>1, 0, -1</td>
</tr>
<tr>
<td></td>
<td>(K^*)</td>
<td>us, d(\bar{s}), s(\bar{d}), s(\bar{u})</td>
<td>1, 0, -1</td>
</tr>
<tr>
<td></td>
<td>(\omega)</td>
<td>(uu + d(\bar{d}))/(\sqrt{2})</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>(\psi)</td>
<td>cc</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>(D^*)</td>
<td>cd, c(\bar{u}), c(\bar{u}), d(\bar{c})</td>
<td>1, 0, -1</td>
</tr>
<tr>
<td></td>
<td>(\Upsilon)</td>
<td>bb</td>
<td>0</td>
</tr>
</tbody>
</table>

### 8. Scattering

Before studying scattering, meaning transitions from one state to another, we need another picture.

#### 8.1. The interaction picture

Back for the moment, to the Schrödinger picture (SP). From the equation (7.12) for the total Lagrangian density, we can write the Hamiltonian (total, not density) as

\[
H := H^S = H^{\frac{1}{2}, 1} = \int (\pi_r \phi^r - \mathcal{L}_0^1 - L_0^\frac{3}{2}) d^3x - \int \mathcal{L}_{\frac{1}{2}, 1}^1 d^3x \quad (8.1)
\]

\[
(H_0^S = H_0) \quad \quad \quad (H_I^S)
\]

which is just
\[ H = H_0 + H_I^S, \] \hspace{1cm} (8.2)

the free part plus the interaction part. In this equation, \( \phi^r \) represents any quantum field. The unitary time evolution operator is then
\[ U(t) = e^{-i(H_0 + H_I^S)t} = U_0(t)U_I(t). \]

Let’s go to a special Heisenberg picture (HP) which is called the interaction picture (IP), one where the unitary transformation uses only the free part of the Hamiltonian:\(^{184}\)
\[ U_0 = e^{-iH_0t}. \] \hspace{1cm} (8.3)

Since we have defined
\[ L^H(t) = U_0^\dagger(t)L^S|U(t), \] \hspace{1cm} (6.18)
the transformation from the SP to the IP is defined by
\[ |\Psi\rangle_I = U_0^\dagger|\Psi\rangle_S \]
as in equation (6.19) for the Heisenberg picture, so that
\[ O^I = U_0^\dagger O^S U_0. \] \hspace{1cm} (8.4)

Then, because \( H_0 \) commutes with itself, the free part of the Hamiltonian \( H_0 = H_0^S \)
\[ H_0^I = U_0^\dagger H_0^S U_0 = H_0, \] \hspace{1cm} (8.5)
but the same is not generally true of the interaction part, \( H_I^I \). So from (8.2) and (8.5), the interaction-picture Hamiltonian is
\[ H_I^I = H_0 + H_I^I. \] \hspace{1cm} (8.6)

This notation can be confusing:
- The subscript \( I \) refers to the interaction energy of the EM field and the Dirac electron;
- the superscript \( I \) indicates the interaction picture.

Remember that
\[ H_0 = H_0^S = H_0^I. \] \hspace{1cm} (8.7)

The equation of motion for an operator in the IP is then
\[ \frac{dO^I}{dt} = \frac{d}{dt}(U_0^\dagger O^S U_0) = ... = -i[H_I^I, O^I] + \frac{\partial O^I}{\partial t}, \] \hspace{1cm} (8.8)
where the last quantity
\[ \frac{\partial O^I}{\partial t} := U_0^\dagger \frac{\partial O^S}{\partial t} U_0 \] \hspace{1cm} (8.9)
is for our purposes always 0. It can also be shown that\(^{185}\)
\[ i\frac{d}{dt}|\Psi\rangle_I = H_I^I|\Psi\rangle_I. \] \hspace{1cm} (8.10)

---

\(^{184}\) This is the version of Klauber, of Lancaster and Blundell and of Schwichtenber PS, but he uses a different method in NNOFT.
\(^{185}\) Klauber, Solution, problem 7.8, 7-5.
The equations of motion for states in the IP depend only on the interaction part of the Hamiltonian.

This is worth repeating:

• The equations of motion (8.8) for operators in the IP depend only on the free part of the Hamiltonian.
• The equations of motion (8.10) for states in the IP depend only on the interaction part of the Hamiltonian.

The first point means that all the results we have studied above for free fields in the HP apply also to operators in the IP. Since that applies to HP field operators like \( \psi \), \( \phi \), or \( A^\mu \), this means that we can use the K-G equation for scalar fields, the Dirac equation for spinors and the Maxwell equation for photons. Not only that, but we can use the free-field

• operator solutions,
• creation, annihilation and number operators,
• observables operators, and
• Feynman propagators.\(^{186}\)

There remains “only” to solve equation (8.10) for the states.

For expectation values,

\[
\frac{d\langle O \rangle}{dt} = i\langle \Psi | \left(-i[H^I, H^F] + \frac{\partial O^I}{\partial t}\right) |\Psi\rangle_I
\]

(8.11)

This is the same equation as that for the expectation value in the SP, even though operators in that picture do not vary in time.

Note that both methods of deriving the IP agree in their results for the time-evolution dependence of the operators and state vectors.

### 8.2. The S-matrix and ordering

We’ll use the word “scattering” rather largely.

Consider the implications of the division of the Hamiltonian into a free part and an interaction part. To calculate the probability that a particular reaction or scattering process takes place, we want the real amplitude between the initial and final states

\[
A = \langle q_1q_2 | p_2 p_1 \rangle_r,
\]

corresponding to the entire Hamiltonian, free plus interaction. But the states we see around us are free ones, corresponding to \( H_0 \). They are the same as the true, or real, states at \( t = \pm \infty \), but not necessarily (or usually) at any other time.

\[
\lim_{p_1 \to -\infty, p_2 \to +\infty} | p_2(t_2) p_1(t_1) \rangle_r = | p_2 p_1 \rangle_0.
\]

(8.12)

Rather than defining one of the two types of states in terms of the other, we define the real amplitude in terms of the free states and the **S-matrix**.\(^{187}\)

\[
A = \langle q_1q_2 | p_2 p_1 \rangle_r := \langle 0 | q_1q_2 \hat{S} | p_2 p_1 \rangle_0.
\]

(8.13)

So to calculate scattering amplitudes, we need the simple states and a way to calculate \( \hat{S} \). We will use the eigenstates of \( H_0 \) and will henceforth drop the 0 subscript on the bras and kets.\(^{188}\)

---

186 Klauber, 194.
187 One more genial idea of John Wheeler.
188 I rather like this introduction to the S matrix, as given by Blundell and Lancaster, 166-7, on which this one is based.
Using the interaction picture, the initial and final states can be taken as free states, so we can construct them from the vacuum state by using the creation and annihilation operators we have already studied.

As for the time-evolution operator, we know that the states in the IP evolve by the interactive part of the Hamiltonian, and so

\[ i \frac{d}{dt}|\Psi(t)\rangle^I = H_I^f |\Psi(t)\rangle^I, \]

which leads to the equation for the unitary time evolution operator \( U(t_2, t_1) \)

\[ i \frac{d}{dt} \hat{U}(t_2, t_1) = H_I^f (t_2) \hat{U}(t_2, t_1). \]  
(8.14)

Unfortunately, since \( H_I^f(t) \) may not commute with itself at different times, the “simple” solution

\[ \hat{U}(t_2, t_1) = e^{-i \int_{t_1}^{t_2} dt \hat{H}_I^f(t)} \]

is shown to be incorrect when this is expanded in a power series, as we must do in order to solve it.\(^ {189} \) Instead, we must use the time-ordered solution

\[ \hat{U}(t_2, t_1) = T[e^{-i \int_{t_1}^{t_2} dt \hat{H}_I^f(t)}], \]  
(8.15)

where the time-ordering operator (It’s not really an operator, but a specification of method) arranges the elements of each term of the power-series expansion of the formula from right to left in order of increasing time.

So now we have learned of two methods of ordering operators:

- **normal ordering** puts creation operators on the left, annihilation operators on the right;
- **time ordering** puts operators in order of increasing time from right to left, earliest to the right, latest to the left.

Merging (8.13) and (8.15) gives us the S-matrix equation

\[ \hat{S} = T[e^{-i \int_{t_1}^{t_2} dt \hat{H}_I^f(t)}]. \]  
(8.16)

### 8.3. Contractions and Wick’s theorem

Equation (8.16) for the S-matrix uses time ordering, but normal ordering is easier to work with, as the annihilation operators on the right destroy a vacuum state, as do creation operators on the left, thus simplifying the equation. Here is where we need contractions, or Wick contractions. A field operator such as those we have found in, for instance equations (6.33) are sums of creation and annihilation parts. Note that we are now using the opposite notation to that of section 6\(^ {190} \), so \( \hat{A}^+ \) now denotes the creation part and \( \hat{A}^- \) the annihilation, which we therefore want on the right in normal ordering. We could consider this for two operators \( \hat{A} \) and \( \hat{B} \) such that

\[ \hat{A} \hat{B} = (\hat{A}^+ + \hat{A}^-)(\hat{B}^+ + \hat{B}^-). \]

where this pair of operators differs from normal ordering \( N[\hat{A} \hat{B}] \), often delimited by colons as \( \hat{A} \hat{B}^{\cdot} \), by the commutator of \( \hat{A} \) and \( \hat{B} \),

\[ \hat{A} \hat{B} = N[\hat{A} \hat{B}] + [\hat{A}^- , \hat{B}^+] \].

Or we could consider the same operator evaluated at different spacetime moments, using the notation of (6.34).

---

189 Schwichtenberg, NNQFT, 388-9.
190 Klauber’s notation.

Symmetry, groups and quantum field theory 91 2021-12-13
\[ \hat{\phi}(x) = \hat{\phi}^+(x) + \hat{\phi}^-(x), \]

where the + and – superscripts now again refer to creation and annihilation terms, respectively.\(^{191}\)

Then

\[ \hat{\phi}(x) \hat{\phi}(y) = N[\hat{\phi}(x) \hat{\phi}(y)] + [\hat{\phi}^-(x), \hat{\phi}^+(y)]. \quad (8.17) \]

In either case, this tells us that we can normal-order operators as long as we include the appropriate commutator terms. It looks interesting because a commutator either is zero or has a value like some multiple of \( \hbar \) and maybe a delta function. In other words, it’s just a number, perhaps complex, called a c-number. Since time ordering of the same two operators gives

\[ T[\hat{A}(x)\hat{B}(y)] = \begin{cases} \hat{A}(x)\hat{B}(y), & x^0 > y^0 < 0 \\ \hat{B}(y)\hat{A}(x), & x^0 < y^0 \geq 0 \end{cases}, \]

\((x^0 = t)\) then the difference between the time-ordered pair and the normal-ordered one

\[ AB = T[\hat{A}(x)\hat{B}(y)] - N[\hat{A}(x)\hat{B}(y)] = \begin{cases} [\hat{A}^-(x), \hat{B}^+(y)], & x^0 > y^0 \\ [\hat{B}^-(y), \hat{A}^+(x)], & x^0 < y^0 \end{cases}, \quad (8.18) \]

is so useful that it is defined to be the contraction or the Wick contraction. The first term in the commutator is time-ordered, since it originates from the time-ordered sequence, with the creation term on the right, as that is the one left over after subtraction of the normal-ordered sequence. In the case of two operators, we can find their contraction by calculating the result of their time-ordered product between \( \langle 0 \rangle \) and \( |0\rangle \), since the normal-ordered part will give zero and the contraction, by (8.18), is simply a commutator, a complex number.\(^{192}\)

\[ \phi(x)\phi(y) = \langle 0 \mid T\phi(x)\phi(y)\rangle |0\rangle. \quad (8.19) \]

Since fermions do not commute but anticommute, a factor of -1 is introduced for each contraction of a pair of fermions.

(Note on notation: A contraction symbol is most often like a square bracket lying on its side, either above or below the characters involved, with its two ends pointing at the operators to be contracted. Not finding this in TexMaths for LibreOffice, I am obliged to settle for an underline, which will not always be adequate, in which case I will be forced to fall back on words. Another possibility, used by Wikipedia, is with bullet superscripts, where pairs are linked by an identical number of bullets:. E.g., \( N[\hat{A}^\bullet \hat{B}^\bullet \hat{C}^\bullet \hat{D}^\bullet] \), where AD is a contracted pair and BC is another. Then \( AB = A^\bullet B^\bullet. \)

Equation (8.18) is a special case of Wick’s theorem, for a pair of operators. The theorem states, more generally, that a time-ordered product of creation and annihilation operators can be expressed as the normal ordered product of the fields plus the normal-ordered sum of all possible contractions of the fields.\(^{193}\)

For an arbitrary number of fields, this is:\(^{194}\)

\[ T\{\phi(x_1)\phi(x_2)…\phi(x_n)\} = N\{\phi(x_1)\phi(x_2)…\phi(x_n)\} + all\ possible\ contractions\}. \]

Note that the entire expression on the right is normal-ordered, including the contractions; i.e., each term is normal ordered. The theorem can be proved by induction.\(^{195}\) The really useful fact here is that when operating on the vacuum, the normal-ordered term will annihilate the state vector, on the right or on the left (\( \langle 0\mid A^\dagger = 0 \)), and so gives zero. So nothing is left but the contractions. Only completely contracted terms will give non-zero results.

---

\(^{191}\) Be aware that some authors, such as Klauber or Peskin and Schroeder, use the + and – superscripts in the opposite sense, with \( \hat{\phi}^+ \) being the annihilation operator. Comprenez qui pourra.

\(^{192}\) Lancaster and Blundell, 172; Schwichtenberg, QFT, 427.

\(^{193}\) Schwichtenberg, op. cit., 428.

\(^{194}\) Lancaster and Blundell, 172; Schwichtenberg, QFT, 428.

\(^{195}\) Peskin and Schroeder, 89-90.
in a transition amplitude between initial and final vacuum states.

An example of a more complicated case would be that of four operators, in which individual contractions of operators as well as of pairs of contractions must be taken into account.

\[
T[\hat{A}\hat{B}\hat{C}\hat{D}] = N[\hat{A}\hat{B}\hat{C}\hat{D}] + N[\hat{A}^*\hat{B}\hat{C}\hat{D}] + N[\hat{A}\hat{B}^*\hat{C}\hat{D}] + N[\hat{A}\hat{B}\hat{C}^*\hat{D}] + N[\hat{A}\hat{B}\hat{C}\hat{D}^*] + N[\hat{A}\hat{B}\hat{C}\hat{D}] + N[\hat{A}^*\hat{B}^*\hat{C}\hat{D}^*] + N[\hat{A}\hat{B}^*\hat{C}\hat{D}^*] + N[\hat{A}\hat{B}^*\hat{C}^*\hat{D}^*] + N[\hat{A}^*\hat{B}\hat{C}\hat{D}^*] + N[\hat{A}^*\hat{B}^*\hat{C}^*\hat{D}^*].
\]

If \( A \) and \( B \) are free scalar fields \( \phi(x) \) and \( \phi^\dagger(y) \), then the quantity

\[
\langle 0|T\{\phi(x)\phi^\dagger(y)\}|0\rangle = \langle 0|\phi(x)\phi^\dagger(y)|0\rangle \equiv i\Delta_F(x - y)
\]

is the **Feynman propagator** of equation (6.48).

Now we can prepare a summary of important contractions for later use.\(^{196}\) (Beware of slightly different notations or factors of \( I \) among authors, especially in the propagator terms.)

\[
\phi(x), \phi(y) = D_F(x, y), \quad \text{the Feynman propagator} \tag{8.20}
\]

\[
\phi(x)|1_k\rangle = \phi(x)a_k^\dagger|0\rangle = e^{-i(kx)}|0\rangle \tag{8.21}
\]

\[
\langle 1_k|\phi(x) = \langle 0|a_k\phi(x) = \langle 0|e^{i(kx)}e^{-i(kx)} \tag{8.22}
\]

\[
\langle 1_q|\cdots|1_k\rangle = \sqrt{2\omega_q}\sqrt{2\omega_k}(2\pi)^3\delta(\vec{q} - \vec{k}) \tag{8.23}
\]

\[
|1_q, 1_k\rangle = 0 \tag{8.24}
\]

\[
\langle 1_q, 1_k\rangle = 0 \tag{8.25}
\]

While we’re at it, let’s write the contraction formula (8.18) in terms of our \( \phi \) variables:

\[
\phi(x), \phi(y) = \begin{cases} 
[\phi_-(x), \phi_+(y)], & x^0 > y^0 < 0 \\
[\phi_-(y), \phi_+(x)], & x^0 < y^0 \geq 0
\end{cases} \tag{8.26}
\]

Last but not least, Wick’s theorem

\[
T\{\phi(x_1)\phi(x_2)\cdots\phi(x_n)\} = N\{\phi(x_1)\phi(x_2)\cdots\phi(x_n) + \text{all possible contractions}\} \tag{8.27}
\]

### 8.4. Putting it all together – the Dyson series

So, now, we have Lagrangians for free scalars, spinors and vectors. Requiring (local) gauge invariance under our favorite Lie groups gives us minimal-coupling interactions, and we can use those to build the S-matrix for particle interactions plus creation and destruction of particles or energy quanta. And we can use operator commutation relations to simplify the search for solutions. So what does that give us?

The time for even the limited amount of detail we have been giving is past. We must accelerate. We will merely summarize the rest of this section.\(^{197}\)

We want to

- start from nothing, the vacuum state,
- build some initial and final states, and

\(^{196}\) Schwichtenberg, QFT, 430-6, 461.

\(^{197}\) Summary based on Schwichtenberg, NNQFT, section 11.7.
investigate the ways in which one can transform into the other.

Starting from the transition amplitude of equation (8.13) and based on the vacuum state, we want to consider
\[ \langle 0 | S | 0 \rangle, \]
where now S must not only create the necessary particles to give us our initial and final states but also represent
the time evolution of the initial state into the final one. The classic example used is the $\phi^4$ theory with scalar
Lagrangian (7.30), i.e.,
\[ \mathcal{L} = \frac{1}{2} (\partial \mu \phi) (\partial^\mu \phi) + \frac{1}{2} \mu^2 (\phi^* \phi) - \frac{1}{4} \lambda^2 (\phi^* \phi)^2, \]  
(7.30)
which leads to the Hamiltonian
\[ H = \int d^3 x \left( \frac{1}{2} (\partial \mu \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{4!} \lambda \phi^4 \right) = H_0 + H_I, \]  
(8.28)
containing a “free” and an interaction part.\(^{198}\) Hopefully, $\lambda$ is a small quantity so we can ignore higher orders of it
in our calculations. We know from equation (8.16) that we must use the time-ordered evolution operator so the
scattering operator, the S-matrix, is
\[ \hat{S} = T[e^{-i \int_{-\infty}^{\infty} dt' \hat{H}_I(t')}], \]
using the interaction part of the Hamiltonian.\(^{199}\) This can be expanded to give the transition amplitudes
\[
A(i \rightarrow f) = \langle f | \hat{S} | i \rangle \\
= \langle f | i \rangle - i \langle f | \int_{-\infty}^{\infty} dt_1 H_I(t_1) | i \rangle \\
- \frac{1}{2!} \langle f | T \left( \int_{-\infty}^{\infty} dt_1 H_I(t_1) \right) \left( \int_{-\infty}^{\infty} dt_2 H_I(t_2) \right) | i \rangle + ... \\
= A^{(0)} + A^{(1)} + A^{(2)} + ..., \]  
(8.29)
This is the Dyson series. Note that we are in the interaction picture (IP), so we use the interaction part of the
Hamiltonian, as was calculated in equation (8.14). Note also the presence of the time-ordering operator $T$ in all
terms from $A^{(1)}$ on.

As an example, let’s consider 2-pion scattering with the $\phi^4$ potential of equation (Error: Reference source not
found). Our initial and final states are then
\[ \langle f | = \langle \pi_{k_3}^0, \pi_{k_4}^0 | \quad \text{and} \quad \ | i \rangle = | \pi_{k_1}^0, \pi_{k_2}^0 \rangle \]
For simplicity, we will consider only initial and final states where $(k_3, k_4) = (k_1, k_2)$.

Various techniques can be used to evaluate the zero-th order term to 1 in the case of equal momenta for the two
pions before and after “interaction”\(^{200}\), otherwise zero. This is the case of nothing happening, and that is not
interesting. So we will skip on to the first-order term.

The formulae (8.20) through (8.27) can be used to evaluate the first and second order terms, which we will
simply cite here.
\[
A^{(1)} = -i \lambda V 
\]  
\(^{198}\) Schwichtenberg, QFT, 407. 
\(^{199}\) Schwichtenberg, QFT, 393. 
\(^{200}\) Leaving out a whole lot here. See Schwichtenberg, op. cit. 409-11.
where V is the volume integrated and in principle will disappear at a later moment.

### 8.5. The final step – Feynman diagrams

Now we have a power series (8.29) to represent the probability amplitudes we want to calculate. Each term is a product of contractions. The evaluation of the terms in the series is facilitated greatly by the use of Feynman diagrams. Here’s how it’s done.

- The total transition amplitude is represented by a power series, each term of which is represented by a diagram.
- Just as each term in the series is a product of contractions, so will each diagram consist of a selection of factors corresponding to the contractions. Their value is given by the so-called Feynman rules, which depend on the interaction type. One set of Feynman rules is just the equations (8.20) - (8.25).

For our examples, Feynman diagrams are constructed according to the **Feynman rules for the \( \phi^4 \) model**, which are resumed in Table 12. The rules give the diagrammatic representation and the amplitude factor for each possible type of contraction.

Each term in the series will include a power of the **coupling constant** for the basic interaction, \( \lambda \) in our \( \phi^4 \) examples. More complicated terms (diagrams) lead to higher powers of the coupling constant (cc). If the cc is small, as for QED where its standard value is approximately \( \alpha = \frac{1}{137} \), higher powers will be so small they can be neglected and this simplifies the calculation. For the strong force, this is complicated by the creation of virtual gluons referred to in section 7.9 which make the cc at a distance seem much larger than 1. Fortunately, at closer distances the cc becomes smaller and calculations are possible.\(^{201}\)

<table>
<thead>
<tr>
<th>Contraction of</th>
<th>represents</th>
<th>Amplitude factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>a field with a particle in the initial state, ( \phi(x)</td>
<td>1_k )</td>
<td>an incoming external line coming from outside and leading to ( x )</td>
</tr>
<tr>
<td>a field with a particle in the final state, ( 1_k</td>
<td>\phi(x) )</td>
<td>an outgoing external line starting at ( x ) and ending outside the diagram</td>
</tr>
<tr>
<td>a field with itself, ( \phi(x) \phi(y) )</td>
<td>an internal line running from ( x ) to ( y )</td>
<td>( D_F(x, y) ) (propagator)</td>
</tr>
<tr>
<td>a point where lines meet</td>
<td>a <strong>vertex</strong>, denoted by a dot</td>
<td>(-i\lambda) for each vertex</td>
</tr>
<tr>
<td>a line entering and leaving without interaction</td>
<td></td>
<td>( \sqrt{2\omega_q \sqrt{2\omega_k} (2\pi)^3 \delta(\vec{q} - \vec{k})} )</td>
</tr>
</tbody>
</table>

Table 12. Feynman rules for \( \phi^4 \) potential.

The first three of these configurations are just restatements of equations (8.20) - (8.25).

It is Wick’s theorem that allows us to write any expression of the form

\[ \langle 0 | T \{ \phi(x_1) \phi(x_2) \ldots \phi(x_n) \} | 0 \rangle \]

\(^{201}\) Griffiths, 70.
as a sum of products of Feynman propagators. Let's go back and consider the four-field case. simplifying the notation some from the second part on.

\[ T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\} = N\{\phi_1\phi_2\phi_3\phi_4 + \phi_1^*\phi_2^*\phi_3\phi_4 + \phi_1\phi_2^*\phi_3^*\phi_4 + \phi_1^*\phi_2\phi_3^*\phi_4^* \\
+ \phi_1\phi_2^*\phi_3\phi_4^* + \phi_1\phi_2\phi_3\phi_4^* + \phi_1\phi_2^*\phi_3^*\phi_4^* + \phi_1^*\phi_2\phi_3^*\phi_4^* \}
\]

When we sandwich this between two vacuum states, with the help of (8.20), the result is

\[ \langle 0|T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\}|0\rangle = D_F(x_1 - x_2)D_F(x_3 - x_4) \\
+D_F(x_1 - x_3)D_F(x_2 - x_4) + D_F(x_1 - x_4)D_F(x_2 - x_3). \] (8.30)

since the normal-ordered term and any un-contracted terms result in 0. In terms of Feynman diagrams, this case gives the following three ways of connecting points 1-4.

![Feynman diagrams for equation (8.30).](image)

More interesting examples include these for two-particle scattering.

![Figure B. Feynman diagrams for s, t and u-channel scattering.](image)

The amplitude matrices for these diagrams, assuming coupling constant g at both vertices, are as follows:

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202 Peskin and Schroeder, 89-90.

Symmetry, groups and quantum field theory
for the s-channel,\
\[ \mathcal{M}_s = (-ig) \frac{i}{(p_1 + p_2)^2 - m^2 + i\epsilon}(-ig) = \frac{-ig^2}{s - m^2 + i\epsilon}, \]

for the t-channel, and\
\[ \mathcal{M}_t = (-ig) \frac{i}{(p_1 - p_3)^2 - m^2 + i\epsilon}(-ig) = \frac{-ig^2}{t - m^2 + i\epsilon}, \]

for the u-channel. The quantities s, t and u are the **Mandelstam variables**. They satisfy\
\[ s + t + u = \sum m_j^2, \]

the sum of the invariant masses of the four particles involved. Note that the s-channel represents an annihilation, whereas the t and u channels are scattering processes.

### 9. Path integral formulation

The path-integral formulation of QFT, usually attributed to Feynman but in fact invented years before by Dirac, takes off from the double-slit experiment. It assumes multiplication not only of the number of slits but even of the screens containing them until there is nothing left but space. We must integrate over all possible paths between the light source and the chosen destination point on the detecting screen in order to find the result of the superposition of all those waves. So the integral is not just over the paths due to the two slits, but over all possible paths in space. Hence the name, "**path integral**".

The amplitude for a particle to start at spacetime point \( q_1, t_1 \) and propagate to \( q_2, t_2 \) can be calculated by using the unitary time-evolution operator:
\[
\langle q_2, t_2 | q_1, t_1 \rangle = \langle q_2 | e^{-iH(t_2-t_1)} | q_1 \rangle.
\]

In order to integrate over all possible paths, we first break the time interval into N segments, each of length
\[
\delta t = \frac{t_2-t_1}{N}.
\]
Since the time-evolution operator is unitary, \( U(t_b - t_a) = U(t_b - t_x)U(t_x - t_a) \), we can use time slicing:
\[
\langle q_2, t_2 | q_1, t_1 \rangle = \langle q_2 | (e^{-iH\delta t})^N | q_1 \rangle,
\]
then insert N identity operators, each of which is curiously called a **fat unity**:
\[
\int dq_n |q_n\rangle \langle q_n| = 1.
\]

Collect the integral signs at the beginning and rename the initial and final states a and b to avoid confusion:
\[
\langle q_b, t_b | q_a, t_a \rangle = \int dq_1 dq_2 ... dq_N \langle q_a | e^{-iH\delta t} | dq_N \rangle ... \langle q_{n+1} | e^{-iH\delta t} | dq_n \rangle ... \langle q_1 | e^{-iH\delta t} | dq_a \rangle,
\]

which contains a product of mini-propagators. Using a classical Hamiltonian like \( H = \frac{\hat{p}^2}{2m} + V(q) \), some substantial but basic calculation leads finally to

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203 This particular explanation is from Blundell and Lancaster, 210-212. I think it is the clearest. See also Robinson, 219-222.

204 Blundell and Lancaster, 211.
\[ \langle q_b, t_b|q_a, t_a \rangle = \int Dq e^{i \int_{t_a}^{t_b} dt L} = \int Dq e^{i S}, \]  \tag{9.1}

where \( L \) is the Lagrangian, \( S \) the action and

\[ Dq = \prod_{i=1}^{\infty} dq_i. \]  \tag{9.2}

Equation (9.1) is a **sum over paths** with each path weighted statistically by \( e^{i S} \). As far as I can tell, this derivation does not depend on use of the commutation relations of second quantization, making the path-integral formulation an **alternative** to second quantization.\(^{205}\)

Note that the result (9.1) is essentially a product of exponentials, each one of which contributes a phase. One can expand the difference between the phase at the minimum of the action and that at a point infinitesimally close to it in a Taylor’s series. If the result is evaluated at a point close to the minimum of the action, its first derivative will disappear, so the difference will be a minimum and the phases almost the same. They then will add constructively and the probability amplitude at this point will be large. So the probability will be greatest at the point of minimum action, in accordance with classical mechanics. Although this may not be important for an individual event, the statistical sum of such events will show a strong probability for the classical result at a minimum of the action.

In order to actually use this formulation to calculate probability amplitudes, we introduce the **generating functional**

\[ Z[J] = \int D\phi \exp \left\{ iS[\phi] + i \int d^4x J(x) \phi(x) \right\}, \]  \tag{9.3}

where

\[ S[\phi] = \int d^4x L \]

and we introduce an **external source** \( J(x) \). Note the square brackets to indicate that \( Z[J] \) is a functional of \( J \), not a function. We also need the **functional derivative**, or **generating functional**, defined by analogy with

\[ \left( \frac{\partial}{\partial x_i} \right) J(x) = \delta^4(x - y) \]  \tag{9.4}

so that

\[ J(y) = \int d^4x \delta(x - y) J(x). \]

Then

\[ -i \frac{\partial Z}{\partial J(x_1)} = \int D\phi \exp \left\{ iS[\phi] + i \int d^4x J(x) \phi(x) \right\} \phi(x_1) \]

brings “down” \( \phi \) evaluated at the point \( x_1 \) and

\[ -i \frac{1}{Z(0)} \frac{\partial Z}{\partial J(x_1)} \bigg|_{J=0} = \frac{\int D\phi \exp \left\{ iS[\phi] \right\} \phi(x_1)}{\int D\phi e^{i \int d^4x L(\phi)}} = \frac{\langle \hat{\phi}(x_1) | \Omega \rangle}{\langle \Omega | \phi(x_1) | \Omega \rangle}, \]

\(^{205}\) Is this true? [?]
where $|\Omega\rangle$ is the vacuum state. Better yet, applying this derivative $n$ times gives

$$(-i)^n \frac{1}{Z(0)} \frac{\partial^n Z}{\partial J(x_1)\ldots \partial J(x_n)} \bigg|_{J=0} = \langle \Omega | T\{\hat{\phi}(x_1)\ldots \hat{\phi}(x_n)\} | \Omega \rangle,$$

where the operators on the right are automatically time-ordered. (What more can we ask?) So the generating functional (functional derivative) can be used to calculate any amplitude. Find it and your problem is solved. Right, as John Cleese would say!

Indeed, for the classical scalar Lagrangian density (6.27), this can be solved and the resulting amplitude is the Feynman propagator of (6.50). Although the result lacks the term $i\epsilon$ in the denominator, this will be recovered later in the boundary conditions for S-matrix calculation, but we won’t go there.

### 10. Renormalization

Anytime there is a loop in a Feynman diagram, there is the possibility of an infinite result. Consider the diagram in Figure 10, a possible Feynman diagram.

![Figure 10. Possible second-approximation term (s-channel)](image)

It is clear that all we can say from conservation of energy is that $k_1 + k_2 = k_3 + k_4$. The quantity $q$ is undetermined and so integrating over all possible values of it goes like

$$\int_0^\infty d^4q \frac{1}{q^4} \sim \ln(q) \bigg|_0^\infty \sim \infty$$

and diverges logarithmically.

The classical way of handling this problem was the method of Feynman, Schwinger, Tomonaga and Dyson, to “renormalize” QFT by canceling so-called infinite counterterms. This works, as “QED is the most precisely tested theory of all time.” The more modern way of doing this is due to Ken Wilson and is called Effective Field Theory, or EFT. This method does not obsolete the older one, as different cases are easier to solve with one or the other.

In EFT, we admit we do not know what is going on above some energy called the cutoff energy and denoted here by $E_*$. Above this energy is the so-called “ultraviolet (UV)” zone of higher energies; below, the “infrared (IR)” zone of lower energies. EFT supposes that it is possible to do physics on the low-energy level without worrying about the higher-energy stuff, since we can’t observe it anyway. In fact, this is what we do all the time, since we really don’t know what is going on up there. This being the case, our integrals probably should not extend into this unknown region just in order to satisfy convenient boundary conditions. Specifically, EFT proposes handling Feynman diagrams with loops, which give infinite results, as sums of diagrams below $E_*$ and then the result is finite – sometimes.

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206 See Schwartz, 262 or Robinson, 225-227.
207 But Schwartz does, 264-266.
208 The loop could just as well be represented as a square.
209 Carroll
Dimensional analysis is helpful for understanding. Assuming \( c = \hbar = 1 \):

\[
E = mc^2 \Rightarrow [E] = [M][V]^2 = [M][L]^2/[T]^2, \quad \text{and} \quad c = 1 = [L]/[T]
\]

along with

\[
\hbar = 1 = [M][L]^2/[T]
\]
give us

\[
[L] = 1/[E] = [T].
\]

Then the scalar field-theory Lagrangian goes like

\[
\frac{K.E.}{\text{space volume}} \sim \frac{[E]}{[L]^3} \sim [E^4] \quad \text{and} \quad \frac{1}{2} \left( \frac{d\phi}{dt} \right)^2 \sim \frac{[\phi]^2}{[T]^2} \sim [\phi]^2[E]^2.
\]

This tells us that \([\phi] \sim [E]\), so \(L \sim [E]^4\). Now expand the interaction \(L\) in powers of \(\phi\),

\[
L \sim c_3\phi^3 + c_4\phi^4 + c_5\phi^5 + \ldots.
\]

If we assume a cutoff energy \(E_*\), we can guess that \(c_3 \sim [E_*], c_4 \sim 1\) and \(c_5 \sim [E_*]^{-1}\). We then work at energies well below \(E_*\), where \(c_5\) should be irrelevant. The terminology is

| \(c_2\phi^2, c_3\phi^3\) | are relevant interactions |
| \(c_4\phi\) | are marginal interactions |
| \(c_5\phi^5, c_6\phi^6 \ldots\) | are irrelevant interactions |

With luck, there will only be a small number of relevant or marginal terms, and this usually turns out to be the case. But what is really advantageous is that the physical results of EFT calculations do not depend on the cutoff. This is because the coupling constants, \(c_n\), evolve with changing \(E_*\). There is a specific functional form for the dependence of the \(c_n\) on \(E_*\). The coupling “constants” – which are not at all constant – are said to “flow” or “run” along a trajectory in the space of theories.

This method of calculating is called the renormalization group, although it is not at all a group in the group-theory sense. Referring to \(E_*\) as \(\Lambda\), “In a finite theory with a UV cutoff \(\Lambda\), physics at energies \(E \ll \Lambda\) is independent of the precise value of \(\Lambda\). Changing \(\Lambda\) changes the couplings in the theory so that observables remain the same.”

There is an excellent analogy which considers firing a cannon … underwater. The cannonball now undergoes drag from the water. Nevertheless, its effective mass can be determined by shaking it to and fro, and this measured value then can be used in calculations, otherwise ignoring the influence of the liquid environment. But the effective mass varies with the velocity, since as the latter approaches 0, the effects of the water become negligible. “In other words, the presence of a medium can introduce a scale-dependent effective mass. We say that the effective mass is ‘renormalized’ by the medium. In quantum physics, every particle moves through a ‘medium’ consisting of the quantum fluctuations of all particles present in the theory. We again take into account this medium by ignoring it but changing the values of our parameters to scale-dependent ‘effective’ values.”

The Wilsonian renormalization group is only one of several methods for taming divergent amplitudes in physics. This is a vast subject and, to my mind, somewhat less than passionately interesting – especially in the details – in spite of its importance. After all, if we can’t calculate it, we don’t really know it – at least not completely.

Let’s look at one aspect of this in somewhat more detail, at the risk of repeating things that have already been

210 Schwartz, 417.
stated.

Often an integral which diverges can be “tamed” by putting a cutoff value on the variable of integration, usually denoted \( \Lambda \). It may even come about that \( \Lambda \) disappears in the final equation, even if we let it go to \( \infty \). The step of imposing a cutoff value is called **regularization**. Often, the integral in question is constructed in an unnatural way, with infinite limits to satisfy some idealized boundary conditions. Also, it is not certain – nor agreed upon by all physicists – that the SM is valid at very high energies.

Consider the divergence discovered in the diagram of Figure 10. We will **regularize** the integral in (10.1) by using a cutoff \( \Lambda \) so the amplitude of order 2 can be written as

\[
\mathcal{M}_2 \sim \lambda^2 \int_0^\Lambda d^4q \frac{1}{q^2 - m^2 + i\epsilon} \frac{1}{(k_1 + k_2 - q)^2 + i\epsilon},
\]

which leads to

\[
\mathcal{M} = -\lambda - \lambda^2 C \ln \frac{s}{\Lambda^2}.
\] (10.2)

where \( s \) is the energy scale \( s = (k_1 + k_2)^2 \) and \( C \) is a constant.\(^{213}\) We then compare the amplitude at two values of the energy scale and find

\[
\mathcal{M}(s_1) - \mathcal{M}(s_2) = -\lambda^2 C \ln \frac{s_2}{s_1}.
\] (10.3)

Now we are getting somewhere. Define the **renormalized coupling constant** \( \lambda_R \) by

\[
\lambda_R \equiv -\mathcal{M} = \lambda + \lambda^2 C \ln \frac{s}{\Lambda^2}.
\] (10.4)

Then expand \( \lambda \) about \( \lambda_R \) using \( \lambda = \lambda_R + a\lambda^2 + ... \) and collect all this to get

\[
\mathcal{M} = -\lambda_R - \lambda^2 C \ln \frac{s}{s_0}.
\] (10.5)

This is great! The renormalized coupling constant \( \lambda_R \) is measurable at a given energy scale \( s_0 \). Then the value of these two parameters can be used to predict the amplitude at any other energy scale \( s \) by (10.4). In fact, the renormalized coupling constant is the only one we can measure.

This result can also be derived from the Lagrangian by assuming an extra, infinite term, supposed of the order of \( \lambda_R^2 \):

\[
\mathcal{L} = -\frac{1}{2} \phi \Box \phi - \frac{\lambda_R}{4!} \phi^4 - \frac{\delta \lambda}{4!} \phi^4,
\] (10.6)

for a real scalar field. Then from (10.2)

\[
\mathcal{M}(s) = -\lambda_R - \delta \lambda - \lambda^2 C \ln \frac{s}{\Lambda^2} + \mathcal{O}(\lambda^4_R).
\]

Choosing

\[
\delta \lambda = -\lambda^2 C \ln \frac{s_0}{\Lambda^2}
\]

takes us right back to equation (10.4). The 3\(^{rd}\) term in the Lagrangian (10.4) is called a **counterterm**. Such terms contribute an infinity to the Lagrangian but drop out when physical quantities are computed.

\(^{212}\) Schwartz, 297.
\(^{213}\) Schartz finds \( C = \frac{1}{32\pi^2} \).
11. Annex: Math symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Means</th>
</tr>
</thead>
<tbody>
<tr>
<td>∀</td>
<td>for all, for each</td>
</tr>
<tr>
<td>∈</td>
<td>in, a member of</td>
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<tr>
<td>∃</td>
<td>there exists</td>
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<tr>
<td>or :</td>
<td>such that</td>
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12. Bibliography


